Normal Measures and Tall Cardinals

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The subject of my lecture is part of a joint project with James Cummings. We begin with a brief review of some basic definitions and facts concerning measurable, tall, and strong cardinals.

Definition 1 *The cardinal* $\kappa > \omega$ *is* measurable *if* κ *carries a* κ*-additive non-principal ultrafilter* μ *(which is called a* measure*).*

μ *is* κ-additive *if for every sequence* $\langle A_i \mid i < \alpha < \kappa \rangle$ *such that* $A_i \in \mu$ *for all* $i < \alpha$, $\bigcap_{i<\alpha} A_i \in \mu$.

μ *is* non-principal *if* $A \in \mu \Longrightarrow |A| = \kappa$.

Assuming the Axiom of Choice (AC), a measurable cardinal κ carries a *normal measure*. We say that the measure μ is *normal* if for every $f : \kappa \to \kappa$ such that $\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in \mu$, there is some $\alpha_0 < \kappa$ such that $\{\alpha \mid f(\alpha) = \alpha_0\} \in \mu$.

Definition 2 *Suppose κ is a cardinal and $\lambda \geq \kappa$ is an arbitrary ordinal. κ is λ* strong *if there is an elementary embedding $j : V \to M$ for M a transitive inner model of V such that κ is the critical point of j and $V_\lambda \subseteq M$. κ is* strong *if κ is λ strong for every ordinal $\lambda \geq \kappa$.*

We give now the following definition due to Hamkins.

Definition 3 *Suppose κ is a cardinal and $\lambda \geq \kappa$ is an arbitrary ordinal. κ is λ* tall *if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\kappa \subseteq M$. κ is* tall *if κ is λ tall for every ordinal $\lambda \geq \kappa$.*

Note that κ is strong $\Longrightarrow$ κ is tall $\Longrightarrow$ κ is measurable.

In his 2009 *MLQ* paper "Tall Cardinals", Hamkins made a systematic study of tall cardinals and established many of their basic properties. He also made the interesting observation that

"strongness is to tallness as supercompactness is to strong compactness"

and established many results that either support this thesis directly or are analogues of conjectures believed true about strongly compact and supercompact cardinals. For instance, Hamkins mentions that Gitik's work yields the equiconsistency of the theories "ZFC + There is a strong cardinal" and "ZFC + There is a tall cardinal", a positive answer to the analogue of the question asking whether the theories "ZFC + There is a supercompact cardinal" and "ZFC + There is a strongly compact cardinal" are equiconsistent. Hamkins also shows that any measurable limit of strong cardinals (or even only of tall cardinals) is also tall, an analogue of Menas' result that any measurable limit of strongly compact cardinals is strongly compact. (It is in addition true that the least cardinal κ which is either a measurable limit of strong cardinals or a measurable limit of tall cardinals is not $\kappa + 2$ strong.) He further shows that it is consistent, relative to the existence of a tall cardinal, for the least tall cardinal to be the least measurable cardinal. This is an analogue of Magidor's famous result that it is consistent, relative to the existence of a strongly compact cardinal, for the least strongly compact cardinal to be the least measurable cardinal.

We now turn to the main topic of this lecture. To motivate what we are about to consider, it follows from a theorem of Solovay that if κ is $\kappa + 2$ strong, then κ is a measurable limit of measurable cardinals which carries 2^{2^κ} many normal measures, the maximal number of normal measures a measurable cardinal can carry. (Thus in particular, if κ is $\kappa + 2$ strong, then κ is not the least measurable cardinal.) This raises the following

Question: Suppose κ is a tall cardinal which is not $\kappa + 2$ strong. How many normal measures is it consistent for κ to carry?

To provide answers to our Question, we will examine the cases where the tall cardinal κ is either the least measurable cardinal or the least measurable limit of strong cardinals. Our results will show that, roughly speaking, the answer to our Question is that the number of normal measures can be an arbitrary cardinal $\delta \le 2^{2^\kappa}$. This is what one might expect for the much more difficult analogous question for strongly compact cardinals. In the interest of time, we will discuss only the situation where $\delta < 2^{2^\kappa}$, since $\delta = 2^{2^\kappa}$ is handled differently and can be achieved by forcing over arbitrary models of ZFC with the appropriate large cardinals. For concreteness, we will concentrate on $\delta = 1, 2$ (with 2 being representative of all cardinals $\delta > 1$, $\delta < 2^{2^\kappa}$)). This allows us to state our theorems as follows:

Theorem 1 *Con(ZFC + There is a strong cardinal) $\Longrightarrow$ Con(ZFC + The least measurable and least tall cardinal coincide + The least tall cardinal carries either exactly 1 or exactly 2 normal measures).*

Theorem 2 *a) Con(ZFC + There is a measurable limit of strong cardinals) $\Longrightarrow$ Con(ZFC + The least measurable limit of strong cardinals carries exactly 1 normal measure).*

b) Con(ZFC + There is a cardinal which has both Mitchell order 2 and is a limit of strong cardinals) $\Longrightarrow$ Con(ZFC + The least measurable limit of strong cardinals carries exactly 2 normal measures).

Note that a measurable cardinal κ is said to have *Mitchell order 2* if κ carries normal measures μ_1, μ_2 such that $\mu_1 \in V^\kappa/\mu_2$.

The proofs of Theorems 1 and 2b) require forcing over the appropriate fine structural inner model. While we don't yet know much about what happens when we force over an arbitrary model of ZFC in order to produce a non-$(\kappa+2)$ strong tall cardinal κ having fewer than the maximal number of normal measures, we do have the following theorem.

Theorem 3 *Suppose $V \vDash$ "ZFC + κ is the least measurable limit of tall cardinals". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + κ is the least measurable limit of tall cardinals + κ carries exactly κ^+ many normal measures".*

The proofs of the above theorems will require the use of different types of forcing iterations. We recall that an *iteration with Easton support* is an iteration $\mathbb{P}$ in which direct limits are taken at regular limit stages, and inverse limits are taken elsewhere. An *iteration with non-stationary support* is an iteration $\mathbb{P}$ in which inverse limits are taken everywhere, but if λ is inaccessible, then for any $p \in \mathbb{P}_\lambda$ (where $\mathbb{P}_\lambda$ is the iteration up to stage λ), $p = \langle p_\alpha \mid \alpha < \lambda \rangle$, $\{\alpha < \lambda \mid p_\alpha$ is non-trivial$\}$ is a non-stationary subset of λ. $\mathbb{P}$ is a *Magidor iteration of Prikry forcing* if each component of $\mathbb{P}$ is (a term for) Prikry forcing over some measurable cardinal, with finite support in the stems, and full support in the (terms for) measure one sets.

Turning now to the proofs of our theorems, to prove Theorem 1, suppose $\overline{V} \models$ "ZFC + There is a strong cardinal". We pass to the appropriate inner model of the form $L[\vec{E}]$ for $\vec{E}$ a coherent sequence of extenders such that $L[\vec{E}] \models$ "κ is strong and no cardinal is strong up to a measurable cardinal" and take this as our ground model V.

For the case $\delta = 1$, let $\mathbb{P}$ be the non-stationary support iteration of length κ which begins by adding a Cohen subset of ω and then adds a non-reflecting stationary set of ordinals of cofinality ω to each V-measurable cardinal $\gamma < \kappa$. Arguments due to Hamkins, combined with arguments of Friedman and Magidor, show that $V^{\mathbb{P}} \models$ "κ is both the least measurable and least tall cardinal + κ carries exactly 1 normal measure".

The case $\delta = 2$ can be proven using a more complicated non-stationary support iteration of length $\kappa + 1$ from the optimal assumption of only one strong cardinal. There is, however, an alternative proof using the non-optimal assumption of a cardinal of Mitchell order 2 which is a limit of strong cardinals that we present instead.

Assume $V = L[\vec{E}]$ for $\vec{E}$ a coherent sequence of extenders is such that $V \models$ "κ is the least cardinal of Mitchell order 2 which is a limit of strong cardinals". Since $V \models$ "κ is a measurable limit of strong cardinals", as was previously mentioned, $V \models$ "κ is a tall cardinal". Standard inner model theory tells us that in V, κ carries exactly 2 normal measures.

Let $\mathbb{P}$ be the Magidor iteration of Prikry forcing of length κ which adds a cofinal ω sequence to each V-measurable cardinal $\gamma < \kappa$. By work of Magidor, $V^{\mathbb{P}} \models$ "κ is the least measurable cardinal". By work of Gitik, $V^{\mathbb{P}} \models$ "κ is a tall cardinal". Because V is a fine structural inner model in which κ has Mitchell order 2 and carries exactly 2 normal measures, by Ben-Neria's analysis of the number of normal measures present after a Magidor iteration of Prikry forcing over such models, $V^{\mathbb{P}} \models$ "κ carries exactly 2 normal measures". Thus, in $V^{\mathbb{P}}$, κ is both the least measurable and least tall cardinal and carries exactly 2 normal measures, as desired. This completes the sketch of the proof of Theorem 1. $\square$

Turning now to the proof of Theorem 2a) (the case $\delta = 1$ of Theorem 2), suppose $\overline{V} \models$ "ZFC + There is a measurable limit of strong cardinals". Once again, we pass to the appropriate inner model of the form $L[\vec{E}]$ for $\vec{E}$ a coherent sequence of extenders such that $L[\vec{E}] \models$ "κ is the least measurable limit of strong cardinals". There is nothing more to do, since in such a fine structural inner model, because κ is the least measurable limit of strong cardinals, κ must carry exactly 1 normal measure.

To prove Theorem 2b) (the case $\delta = 2$ of Theorem 2), suppose $\overline{V} \models$ "ZFC + There is a cardinal which has both Mitchell order 2 and is a limit of strong cardinals". As before, we pass to the appropriate inner model of the form $L[\vec{E}]$ for $\vec{E}$ a coherent sequence of extenders such that $L[\vec{E}] \models$ "κ is the least cardinal of Mitchell order 2 which is a limit of strong cardinals" and take this as our ground model V. As we have already observed, in V, κ carries exactly 2 normal measures.

We are now ready to describe informally the partial ordering $\mathbb{P}$ used in the proof of Theorem 2b). $\mathbb{P}$ is the Magidor iteration of Prikry forcing of length κ which either adds a cofinal ω sequence to each $\gamma < \kappa$ which is a V-measurable limit of strong cardinals, or forces each V-strong cardinal $\gamma < \kappa$ to be indestructible under Magidor iterations of Prikry forcing. In particular, for each V-strong cardinal $\gamma < \kappa$, it is possible to write $\mathbb{P} = \mathbb{P}_\gamma * \dot{\mathbb{Q}}$, where $\Vdash_{\mathbb{P}_\gamma}$ "γ is a strong cardinal which remains strong after doing a Magidor iteration of Prikry forcing". It therefore follows that since $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{\mathbb{Q}}$ is a Magidor iteration of Prikry forcing", $V^{\mathbb{P}_\gamma * \dot{\mathbb{Q}}} = V^{\mathbb{P}} \vDash$ "γ is a strong cardinal".

To see that $V^{\mathbb{P}}$ is as desired, we first note that by work of Magidor, $V^{\mathbb{P}} \vDash$ "κ is a measurable cardinal". Once again, Ben-Neria's analysis shows that $V^{\mathbb{P}} \vDash$ "κ carries exactly 2 normal measures". By the definition of $\mathbb{P}$, in $V^{\mathbb{P}}$, all V-measurable limits of strong cardinals below κ now have cofinality ω. In addition, as we just observed, all V-strong cardinals below κ are preserved to $V^{\mathbb{P}}$. Since Magidor's work shows that forcing with $\mathbb{P}$ creates no new measurable cardinals, it therefore follows that $V^{\mathbb{P}} \vDash$ "κ is the least measurable limit of V-strong cardinals, all of which remain strong". However, since set forcing over an $L[\vec{E}]$ model can't create any new strong cardinals, $V^{\mathbb{P}} \vDash$ "κ is the least measurable limit of strong cardinals". $\square$

Turning now to the proof of Theorem 3, suppose $V \models$ "ZFC + κ is the least measurable limit of tall cardinals". For any regular cardinal γ, let $\text{Add}(\gamma, 1)$ be the partial ordering for adding a single Cohen subset of γ. Let $A = \{\gamma < \kappa \mid$ For some tall cardinal $\delta < \kappa$, γ is the least inaccessible cardinal greater than $\delta\}$. Define $\mathbb{Q} = \text{Add}(\omega, 1) * \dot{\text{Add}}(\kappa^+, 1) * \dot{\text{Add}}(\kappa^{++}, 1) * \dot{\mathbb{Q}}^*$, where $\dot{\mathbb{Q}}^*$ is a term for the Easton support iteration of length κ which adds a Cohen subset to each $\gamma \in A$. It is a theorem of Hamkins that any tall cardinal δ has its tallness indestructible under (δ, ∞)-distributive forcing. This, combined with Hamkins' Gap Forcing Theorem, the fact that κ is the least measurable limit of tall cardinals in V, the closure properties of $\text{Add}(\kappa^+, 1) * \dot{\text{Add}}(\kappa^{++}, 1)$, the Lévy-Solovay Theorem for tall and measurable cardinals (i.e., that tall and measurable cardinals are preserved under small forcing), and standard arguments allow us to conclude that $V^{\mathbb{Q}} \models$ "$2^\kappa = \kappa^+$, $2^{\kappa^+} = \kappa^{++}$, κ is the least measurable limit of tall cardinals, and κ carries exactly $2^{2^\kappa} = \kappa^{++}$ many normal measures". With an abuse of notation, we relabel $V^{\mathbb{Q}}$ as V and take V as our ground model.

Working in the new V, let $\mathbb{P} = \mathrm{Add}(\omega, 1) * \dot{\mathrm{C}}\mathrm{oll}(\kappa^+, \kappa^{++})$, where $\dot{\mathrm{C}}\mathrm{oll}(\kappa^+, \kappa^{++})$ is a term for the Lévy collapse of κ^{++} to κ^+. By the Lévy-Solovay Theorem for tall cardinals, Hamkins' Gap Forcing Theorem, the aforementioned indestructibility properties of tall cardinals, and the closure properties of $\mathrm{Coll}(\kappa^+, \kappa^{++})$, $V^{\mathbb{P}} \models$ "κ is the least measurable limit of tall cardinals". By an argument of Cummings, $V^{\mathbb{P}} \models$ "κ carries exactly κ^+ many normal measures". $V^{\mathbb{P}}$ is thus as desired. $\square$

We conclude by asking the general question of how many normal measures a non-$(\kappa+2)$ strong tall cardinal κ can carry when doing set forcing over an arbitrary model of ZFC. This includes models in which GCH fails, models containing very large cardinals such as supercompact cardinals, huge cardinals, etc. We conjecture that this can be any cardinal $\delta \leq 2^{2^\kappa}$, and that ground model large cardinals can be preserved to the generic extension.

Thank you all very much for your attention!