Learnability Thesis, FM–representability and Turing Ideals

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Outline

Church's Thesis

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- $\mathcal{IC}$ - the class of all intuitively computable sets of natural numbers.

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- Δ^0_1 - the class of all recursive sets.

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Thesis (Church's Thesis)

$\mathcal{IC} = \Delta^0_1$.

Intuitive Learnability

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Definition (intuitive learnability)

A decision problem is intuitively learnable if there is an intuitive algorithm that for each example of the problem produces a finite sequence of yeses and nos such that the last answer in the sequence is correct.

Algorithmic Learnability

Mathematical notion of learnability is due to Gold and Putnam:

Definition (Algorithmic Learnability)

Let $A \subseteq \mathbb{N}$. Say that A is algorithmically learnable iff there is a total computable function $g : \mathbb{N}^2 \to \{0, 1\}$ such that for all $x \in \mathbb{N}$:
$\lim_{t\to\infty} g(x, t) = 1 \Leftrightarrow x \in A$ *and* $\lim_{t\to\infty} g(x, t) = 0 \Leftrightarrow x \notin A$.

The notion of algorithmic learnability is one of the equivalent formulations of the concept of methods proceeding by mind-changes and stabilizing on every input.

Finite Model Domains

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Definition (*FM*-domain)

Let $R \subseteq \omega^r$ *be an arithmetical relation. Then by* $R^{(n)}$ *we denote* $R \cap \{0, 1, ..., n\}$. *For any model* $\mathcal{A}$ *over the signature* $\sigma = (R_1, ..., R_k)$ *we define the FM-domain of* $\mathcal{A}$ *as follows:* $FM(\mathcal{A}) = \{\mathcal{A}_n : \; n = 1, 2, ...\}$, *where* $\mathcal{A}_n = (\{0, 1, ..., n\}, R_1^{(n)}, ..., R_k^{(n)})$.

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Definition ($FM(\mathbb{N})$)

$FM(\mathbb{N}) = \{\mathbb{N}_n : \ n = 1, 2, ...\}$, *where*
$\mathbb{N}_n = (\{0, 1, ..., n\}, +^{(n)}, \times^{(n)}, 0^{(n)}, s^{(n)}, <^{(n)})$.

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Definition (Satisfaction in sufficiently large models)

For any formula $\varphi(x_1, \ldots, x_r)$ and $a_1, \ldots, a_r \in \omega$ we say that φ is sl-satisfied in $\mathrm{FM}(\mathcal{A})$ by $a_1, \ldots, a_r$ ($\mathrm{FM}(\mathcal{A}) \models_{sl} \varphi(a_1, \ldots, a_r)$) if and only if

$$\exists m \forall k \ (k \geq m \ \Rightarrow \ \mathcal{A}_k \models \varphi(a_1, \ldots, a_r)).$$

SL-semantics

More generally we could say that for a given class $\mathcal{K}$ of finite models

$$sl(\mathcal{K}) = \{\varphi \in Sent_{\mathcal{L}_\sigma} : \exists n \forall \mathcal{M} \in \mathcal{K} \ (card(\mathcal{M}) \geq n \ \Rightarrow \ \mathcal{M} \models \varphi)\}.$$

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Definition

A set of sentences Δ sl-entails a formula φ, denoted by: $\Delta \models_{sl} \varphi$ if and only if for any given class $\mathcal{K}$ of finite models we have:

$$if \quad \mathcal{K} \models_{sl} \Delta, \quad then \quad \mathcal{K} \models_{sl} \varphi.$$

Logic of (sufficiently large) finite models

When a vocabulary σ is fixed then we define sl-logic as $L_{sl} := sl(\mathbf{MOD}_\sigma)$.

The logic of finite models is defined as $L_{fin} := th(\mathbf{MOD}_\sigma)$.

Let A_{inf} denote the family of sentences $\{\varphi_n : n \geq 2\}$, where φ_n is the following:

$$\exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j.$$

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Let $A_{sl} = L_{fin} \cup A_{inf}$. Then we have the following (all due to M. Mostowski):

Theorem

1. *L_{sl} is the set of all first-order consequences of A_{sl}.*
2. *A_{sl} is Π^0_1-complete.*
3. *L_{sl} is Σ^0_2-complete.*
4. *Completeness: For any set of σ-sentences T containing A_{sl} and any σ-sentence φ we have:*

$$T \vdash \varphi \Leftrightarrow T \models_{sl} \varphi.$$

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We say that the relation $R \subseteq \omega^r$ is FM-represented in $\mathrm{FM}(\mathcal{A})$ by a formula $\varphi(x_1, \ldots, x_r)$ if and only if for each $a_1, \ldots, a_r \in \omega$ both of the following conditions hold:

1. $R(a_1, \ldots, a_r)$ *if and only if* $\mathrm{FM}(\mathcal{A}) \models_{sl} \varphi(a_1, \ldots, a_r)$.
2. $\neg R(a_1, \ldots, a_r)$ *if and only if* $\mathrm{FM}(\mathcal{A}) \models_{sl} \neg\varphi(a_1, \ldots, a_r)$.

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If a relation is FM-representable in $\mathrm{FM}(\mathbb{N})$ *we say that it is* ***FM-representable***.

Limit Lemma

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5. *R is FM-representable.*

Interlude: FM-definability and hierarchies of functions

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Definition

R is FM-definable via f if there is an arithmetical formula $\varphi(x)$ such that for any $n \in \omega$, $\varphi(x)$ defines $R \upharpoonright n$ in $\mathbb{N}_m$, for any $m \geq f(n)$.

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Definition

The class of functions FM-definable via f is denoted by $FM(f)$. Let X be the class of partial functions. Let $Tot(X)$ denote the maximal class $Y \subseteq X$ consisting of total functions. We define $FM(X) = \bigcup_{f \in Tot(X)} FM(f)$.

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Proposition

For any relation R the following are equivalent:

- *there exists a function f such that R is FM-definable via f*
- *there exists a function $g \leq \mathbf{0}'$ such that R is FM-definable via g.*

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Question

Let $\mathbf{a}, \mathbf{b}$ be Turing degrees. Does the following equivalence hold:

$$\mathbf{a} <_T \mathbf{b} \Leftrightarrow \mathbf{a} <_M \mathbf{b}?$$

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Let F, G *be classes of total functions such that* $F \simeq G$*. Then* $FM(F) = FM(G)$*.*

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Proof.

Assume $\mathbf{a} <_T \mathbf{b}$. For the sake of contradiction, assume $\mathbf{a} \not<_M \mathbf{b}$ which means that $\mathbf{a} \not\leq_M \mathbf{b}$ or $\mathbf{b} \leq_M \mathbf{a}$. However, by Lemma 2, $\mathbf{a} <_T \mathbf{b}$ implies $\mathbf{a} \leq_M \mathbf{b}$. So we have $\mathbf{b} \leq_M \mathbf{a}$. But then, by Lemma 1, we get $FM(Tot(\mathbf{a})) = FM(Tot(\mathbf{b}))$. This gives $\mathbf{a} = \mathbf{b}$. □

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Question

$\mathbf{a} <_M \mathbf{b} \Rightarrow \mathbf{a} <_T \mathbf{b}$?

Interlude: Witnessing functions

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Definition (Witnessing function)

Let g be a total function and A_s a recursive approximation of $A = lim_s A_s \in \Delta^0_2$. We say g is a ***witnessing function for*** *A* ***wrt*** *A_s if for every $x \in \mathbb{N}$ we have*

$$x \in A \Leftrightarrow \forall s \geq g(x)\, A_s(x) = A(x)$$

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We say g *is a* ***minimal witnessing function for*** A ***wrt*** A_s, *if for every witnessing function* g' *for* A *wrt.* A_s *we have* $\forall x\, g(x) \leq g'(x)$.

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We say g is a ***minimal witnessing function for*** *A* ***wrt*** *A_s, if for every witnessing function g' for A wrt. A_s we have $\forall\, x\, g(x) \leq g'(x)$.*
We say g is a ***(minimal) witnessing function*** *if there is some A and a recursive approximation A_s such that g is a (minimal) witnessing function for A wrt. A_s.*

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Observe that if g is a minimal witnessing function for $A = lim_s A_s$ and g' is any witnessing function for it, then $g \leq_T g'$.

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We may ask what are degrees of minimal witnessing functions, e.g. is every set $\leq_T \mathbf{0}'$ Turing-equivalent to its minimal witnessing function?

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Theorem

Every minimal witnessing function is of r.e. degree.

Hence, for $A \in \Delta^0_2$ which is not of r.e. degree and for its minimal witnessing function g, we cannot have $deg(A) = deg(g)$.

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Hence, for $A \in \Delta^0_2$ which is not of r.e. degree and for its minimal witnessing function g, we cannot have $deg(A) = deg(g)$.

However:

Theorem

Let $A \in \Delta^0_2$. If A is weakly 1-r.e. then A has a minimal witnessing function g such that $deg(A) = deg(g)$.

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There exist Turing-incomparable functions $f, g < \mathbf{0}'$ *witnessing the same non-recursive set.*

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Every r.e. degree contains a minimal witnessing function.

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There exist Turing-incomparable functions $f, g < \mathbf{0}'$ witnessing the same non-recursive set.

Proposition

There exists a non-recursive total function $\leq \mathbf{0}'$ which does not witness any set $A \in \Delta^0_2 - \mathbf{0}$.
(in fact: let f be any non-recursive function $\leq \mathbf{0}'$, whose values are recursively bounded. Then f does not witness any set $A \in \Delta^0_2 - \mathbf{0}$.)

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Is every total function $\leq \mathbf{0}'$*, that is not recursively bounded, a witnessing function for some non-recursive set?*

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Question

Let $\mathbf{a} < \mathbf{b} \leq \mathbf{0}'$ *and assume* $\mathbf{b}$ *is r.e. Does every* $A \in \mathbf{a}$ *has a minimal witnessing function in* $\mathbf{b}$*?*

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Thesis (Learnability Thesis)

$\mathcal{IL} = \Delta^0_2$.

Church's Thesis and Learnability Thesis

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The Church's Thesis entails the Learnability Thesis.

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Proof.

Assume the Church's Thesis. ($\Delta^0_2 \subseteq \mathcal{IL}$) Let $A \in \Delta^0_2$. Let $g : \omega^2 \to \{0, 1\}$ be algorithmically learnable. By the Church's Thesis, g is an intuitively computable total function. Devise an intuitively learnable infinite procedure for A: let $x \in \omega$. Set $t = 0$. In infinite loop do: intuitively compute $g(t, x)$, output the result in case it differs from the result obtained previously, increment t. This shows $A \in \mathcal{IL}$. □

Church's Thesis and Learnability Thesis

Proposition

The Church's Thesis entails the Learnability Thesis.

Proof.

Assume the Church's Thesis. ($\mathcal{IL} \subseteq \Delta^0_2$) Let $A \in \mathcal{IL}$. Then there is an intuitive algorithm, say G, *learning* A. Devise an intuitive algorithm G' that takes (t, x) as input and returns the last answer generated by G on input x up to t steps of intuitive computation. By the Church's Thesis, the function intuitively computed by G' is recursive. Let $g(t, x)$ be that function. Clearly, g is total and satisfies the definition of algorthmic learnability. Hence, by the Limit Lemma, A is Δ^0_2. □

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Proof.

Assume the Church's Thesis. ($\mathcal{IL} \subseteq \Delta^0_2$) Let $A \in \mathcal{IL}$. Then there is an intuitive algorithm, say G, *learning* A. Devise an intuitive algorithm G' that takes (t, x) as input and returns the last answer generated by G on input x up to t steps of intuitive computation. By the Church's Thesis, the function intuitively computed by G' is recursive. Let $g(t, x)$ be that function. Clearly, g is total and satisfies the definition of algorthmic learnability. Hence, by the Limit Lemma, A is Δ^0_2. □

Question: what about the other direction?

Testing formulae

Testing formulae

Definition (**Testing formula**)

Let $R \subseteq \omega^n$ *and* $\varphi(x_1, \ldots, x_n)$ *be a formula. A formula* $\psi(x_1, \ldots, x_n)$ *is a testing fromula for* $\varphi(x_1, \ldots, x_n)$ *and* R *if:*

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- *for each $a_1, \ldots, a_n \in \omega$ there is $n_0 \in \omega$ such that for each finite model M, $M \models \psi(a_1, \ldots, a_n)$ if and only if $|M| \geq n_0$,*

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- *for each $a_1, \ldots, a_n \in \omega$ and each finite model M, if $M \models \psi(a_1, \ldots, a_n)$, then $R(a_1, \ldots, a_n)$ if and only if $M \models \varphi(a_1, \ldots, a_n)$.*

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The conditions defining the notion of testing formula for φ and R may be read as an explication of the concept of *knowing the answer (and achieving the answer effectively)* to the query of the form: *is a tuple $a_1, \ldots, a_n$ in the relation R?*

Testing formulae

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Theorem (Mostowski)

Let $R \subseteq \omega^n$. R is decidable if and only if there are formulae $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$ such that $\psi(x_1, \ldots, x_n)$ is a testing fromula for $\varphi(x_1, \ldots, x_n)$ and R.

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Theorem (M. Mostowski)

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Proof

Fix $R \subseteq \omega^n$.($\Rightarrow$) Let $T(e, x_1, \ldots, x_n, c)$ be the Kleene predicate: c is the computation of the algorithm e on input $x_1, \ldots, x_n$. Let $U(c, y)$ mean that a computation with code c accepts if $y = 1$ or rejects if $y = 0$. Suppose that R is decidable by an algorithm with the code e. We define:

$$\psi(x_1, \ldots, x_n) = \exists c\, T(e, x_1, \ldots, x_n, c),$$

$$\varphi(x_1, \ldots, x_n) = \exists c\, (T(e, x_1, \ldots, x_n, c) \wedge U(c, 1)).$$

Testing formulae

Proof ...

Fix $\overline{a} = a_1, \ldots, a_n \in \omega$. We show that ψ is a testing formula for φ and R.
We have $\mathbb{N} \models \exists c\, T(e, \overline{a}, c)$ thus for some $n_0 \in \omega$ it holds that $\mathbb{N} \models T(e, \overline{a}, n_0)$. Since the computation of e on $\overline{a}$ is unique, so is n_0.
Therefore for $m \in \omega$, $\mathbb{N}_m \models \psi(\overline{a})$ if and only if $m \geq n_0$.
Now fix $m \in \omega$ such that $\mathbb{N}_m \models \psi(\overline{a})$. Let $n_0 \in \omega$ be such that $\mathbb{N} \models T(e, \overline{a}, n_0)$. Then for every $m \geq n_0$ it holds that $\mathbb{N}_m \models T(e, \overline{a}, n_0)$. If $R(\overline{a})$, then $\mathbb{N} \models U(n_0, 1)$ and $\mathbb{N}_m \models \varphi(\overline{a})$. On the other hand if $\neg R(\overline{a})$, then $\mathbb{N} \models U(n_0, 0)$ and $\mathbb{N}_m \models \neg\varphi(\overline{a})$.
Therefore $\psi(x_1, \ldots, x_n)$ is a testing formula for φ and R.

Testing formulae

Proof......

($\Leftarrow$) Let $\psi(x_1, \ldots, x_n)$ be a testing formula for $\varphi(x_1, \ldots, x_n)$ and R. The algorithm deciding R is the following.

input: $a_1, \ldots, a_n \in \omega$

output: truth value of $R(a_1, \ldots, a_n)$

$i := 0$

while $\mathbb{N}_i \not\models \psi(a_1, \ldots, a_n)$

$i := i + 1$

end while.

return: truth value of $\mathbb{N}_i \models \varphi(a_1, \ldots, a_n)$

The algorithm implicitly uses subroutines to compute $i \mapsto \ulcorner \mathbb{N}_i \urcorner$ and $\mathbb{N}_i \models \alpha$ which are both recursive. It also always halts since $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and R. This ends the proof.

Why testing formulae?

Mostowski proved that using testing formulae enables us to prove the implication: $\mathcal{IL} = \Delta^0_2 \Rightarrow \mathcal{IC} = \Delta^0_1$, if taken together with certain additional assumptions.

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1. There is a recursive enumeration of finite models,
2. Every finite model $\mathbb{N}_m$ has a recursive satisfaction relation.

The main assumptions of Mostowski's argument taken together with the FM-representability theorem are actually equivalent to a version of the Learnability Thesis. Why?

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In particular, such semantics gives us a class of formulae *decidable in the limit*. Such formulae express exactly intuitively learnable concepts. By the FM-representability theorem the set of such concepts is identical to the set of Δ^0_2 relations.

Learnability Thesis and Church's Thesis

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Theorem

Let $(\mathbb{N}, A)$ be any σ'-model, $R \subseteq \omega^n$. R is decidable in A if and only if there are σ'-formulae $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$ such that $\psi(x_1, \ldots, x_n)$ is a testing formula in $\mathrm{FM}((\mathbb{N}, A))$ for $\varphi(x_1, \ldots, x_n)$ and R.

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Taking $\mathrm{FM}(\mathbb{N})$ as our formal model is aimed at distinguishing exactly those properties that are essential for performing intuitive computations.

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$$P(\bar{a}) \equiv \exists x \forall y\, R(x, y, \bar{a}),$$

$$P(\bar{a}) \equiv \forall x \exists y\, S(x, y, \bar{a}),$$

for some recursive in A predicates R and S.

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Proof.

The theorem is obvious by the relativisation of the Limit Lemma. □

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Let A be a low set. Then $\Delta_2^A = \Delta_2^0$.

Proof.

Fix a low set A. and fix a Δ_2^A relation P. Then for some recursive in A predicates R and S we have: $P(\bar{a}) \equiv \exists x \underbrace{\forall y\, R(x, y, \bar{a})}_{\leq \deg(A)'}$ and $P(\bar{a}) \equiv \forall x \underbrace{\exists y\, S(x, y, \bar{a})}_{\leq \deg(A)'}$. Since A is low, $\deg(A)' = \mathbf{0}'$. Therefore P is recursive in $\mathbf{0}'$ and thus, by the Limit Lemma, P is Δ_2^0. □

Lowness

Corollary

Let A be a low set and $R \subseteq \omega^n$. The following are equivalent:

- *R is FM-representable in* $\mathrm{FM}((\mathbb{N}, A))$,
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By the Corollary, adding any low set A to the FM-domain does not affect the class of FM-representable relations and therefore the Learnability Thesis itself.

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We are ready to prove our main theorem:

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Therefore in such model the Church's Thesis fails.
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We have shown that there is an interpretation of $\mathcal{IC}$ such that $\mathcal{IC} \neq \Delta^0_1$ and $\mathcal{IL} = \Delta^0_2$.

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We have shown that there is an interpretation of $\mathcal{IC}$ such that $\mathcal{IC} \neq \Delta^0_1$ and $\mathcal{IL} = \Delta^0_2$. Therefore the Learnability Thesis does not entail Church's Thesis. □

Other interpretations of $\mathcal{IL}$

Definition (Turing ideal)

A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a Turing ideal if for all $X, Y \subseteq \omega$:

- $Y \in \mathcal{A} \wedge X \leq_T Y \Rightarrow X \in \mathcal{A}$,
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Let $\{A_i\}_{i\in\omega}$ be such that $\bigcup_{i\in\omega}\{A_i\}$ is a countable Turing ideal of low sets. Then R is $FM(\mathbb{N}, \{A_i\}_{i\in\omega})$-representable iff R is Δ^0_2.

Conclusion I

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Setting $\mathcal{IL} := \mathbf{0} \cup \mathcal{A}$, where $\mathcal{A}$ is any Turing ideal of low sets, preserves the Learnability Thesis.

Diversity of interpretations of $\mathcal{IC}$

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It is also known by Sacks density theorem that between every pair of recursively enumerable sets A, B such that $A <_T B$ there are incomparable recursively enumerable sets C, D i.e. $A <_T C <_T B$, $A <_T D <_T B$ and $C \perp D$.

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In fact we can have that $C \oplus D \equiv_T B$.

Diversity of interpretations of $\mathcal{IC}$

Therefore let $A = A_\varepsilon$ and B be recursively enumerable sets.

Iterating the result by Sacks we can obtain a full binary tree of degrees below B such that for every $\sigma \in \{0,1\}^*$ it holds that $A_{\sigma 0} \oplus A_{\sigma 1} \equiv_T B$ and $A_{\sigma 0} \perp A_{\sigma 1}$.

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Thus each infinite branch encodes an increasing sequence of recursively enumerable sets that generates a Turing ideal. If on the start we take B low, then every A_σ is low and the ideals generated by infinite branches consist of low degrees only.

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Thus each infinite branch encodes an increasing sequence of recursively enumerable sets that generates a Turing ideal. If on the start we take B low, then every A_σ is low and the ideals generated by infinite branches consist of low degrees only.
It follows that there is a continuum of Turing ideals of low sets, therefore the choice for the interpretation of $\mathcal{IC}$ is indeed wide.

Conclusions II

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If we admit certain non-recursive but intuitively computable relations (namely: some low relations) we are able to consider expanded FM-domains. On the other hand, by the Corollary, relations FM-representable in such FM-dommains are still Δ^0_2.

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The choice of interpretations of $\mathcal{IC}$ is actualy very wide.

Thank you