Learnability Thesis, FM–representability and Turing Ideals

Michał Tomasz Godziszewski
joint work with:
Marek Czarnecki, Dariusz Kalociński

University of Warsaw

CUNY Logic Workshop 2016
GC CUNY
October 28 2016
Outline

1. Learnability and Computability
2. Learnability and Potential Infinity
3. Learnability and Church’s Thesis
Church’s Thesis

Church’s Thesis:

\[ \text{IC} = \Delta^0_1. \]
Church’s Thesis

- $\mathcal{IC}$ - the class of all intuitively computable sets of natural numbers.
Church’s Thesis

- $\mathcal{IC}$ - the class of all intuitively computable sets of natural numbers.
- $\Delta^0_1$ - the class of all recursive sets.
Church’s Thesis

- $\mathcal{IC}$ - the class of all intuitively computable sets of natural numbers.
- $\Delta^0_1$ - the class of all recursive sets.

Church’s Thesis:
Church’s Thesis

- $\mathcal{IC}$ - the class of all intuitively computable sets of natural numbers.
- $\Delta^0_1$ - the class of all recursive sets.

Church’s Thesis:

**Thesis (Church’s Thesis)**

$\mathcal{IC} = \Delta^0_1$. 
Intuitive Learnability

A decision problem is intuitively learnable if there is an intuitive algorithm that for each example of the problem produces a finite sequence of yeses and nos such that the last answer in the sequence is correct.
Intuitive Learnability

Definition (intuitive learnability)

A decision problem is intuitively learnable if there is an intuitive algorithm that for each example of the problem produces a finite sequence of yeses and nos such that the last answer in the sequence is correct.
Algorithmic Learnability

Mathematical notion of learnability is due to Gold and Putnam:

Definition (Algorithmic Learnability)

Let \( A \subseteq \mathbb{N} \). Say that \( A \) is algorithmically learnable iff there is a total computable function \( g : \mathbb{N}^2 \to \{0, 1\} \) such that for all \( x \in \mathbb{N} \):

\[
\lim_{t \to \infty} g(x, t) = 1 \iff x \in A \quad \text{and} \quad \lim_{t \to \infty} g(x, t) = 0 \iff x \notin A.
\]

The notion of algorithmic learnability is one of the equivalent formulations of the concept of methods proceeding by mind-changes and stabilizing on every input.
Finite Model Domains

Definition (FM-domain)

Let $R \subseteq \omega^r$ be an arithmetical relation. Then by $R(n)$ we denote $R \cap \{0, 1, \ldots, n\}$. For any model $A$ over the signature $\sigma = (R_1, \ldots, R_k)$ we define the FM-domain of $A$ as follows: $FM(A) = \{A_n : n = 1, 2, \ldots\}$, where $A_n = (\{0, 1, \ldots, n\}, R_1(n), \ldots, R_k(n))$.

Definition (FM($N$))

$FM(N) = \{N_n : n = 1, 2, \ldots\}$, where $N_n = (\{0, 1, \ldots, n\}, + (n), \times (n), 0 (n), s (n), < (n))$. 

Czarnecki, Godziszewski, Kalociński
Learnability and Church’s Thesis
CUNY LW 2016
**Finite Model Domains**

**Definition (FM-domain)**

Let $R \subseteq \mathbb{N}^r$ be an arithmetical relation. Then by $R^{(n)}$ we denote $R \cap \{0, 1, \ldots, n\}$. For any model $A$ over the signature $\sigma = (R_1, \ldots, R_k)$ we define the FM-domain of $A$ as follows: $FM(A) = \{A_n : n = 1, 2, \ldots\}$, where $A_n = (\{0, 1, \ldots, n\}, R_1^{(n)}, \ldots, R_k^{(n)})$. 

**Definition (FM($N$))**

$FM(N) = \{N_n : n = 1, 2, \ldots\}$, where $N_n = (\{0, 1, \ldots, n\}, +^{(n)}, \times^{(n)}, 0^{(n)}, s^{(n)}, <^{(n)})$. 

Czarnecki, Godziszewski, Kalociński

Learnability and Church’s Thesis

CUNY LW 2016
Definition (FM-domain)

Let \( R \subseteq \omega^r \) be an arithmetical relation. Then by \( R^{(n)} \) we denote \( R \cap \{0, 1, \ldots, n\} \). For any model \( A \) over the signature \( \sigma = (R_1, \ldots, R_k) \) we define the FM-domain of \( A \) as follows: \( FM(A) = \{A_n : n = 1, 2, \ldots\} \), where \( A_n = (\{0, 1, \ldots, n\}, R_1^{(n)}, \ldots, R_k^{(n)}) \).

Definition (FM(\( \mathbb{N} \)))

\( FM(\mathbb{N}) = \{\mathbb{N}_n : n = 1, 2, \ldots\} \), where \( \mathbb{N}_n = (\{0, 1, \ldots, n\}, +^{(n)}, \times^{(n)}, 0^{(n)}, s^{(n)}, <^{(n)}) \).
The notion of algorithmic learnability is one of the equivalent formulations of the concept of methods proceeding by mind-changes and stabilizing on every input. One of them is the notion of FM-representability proposed by M. Mostowski.

Primary motivation: computational foundations of mathematics and the search for the semantics under which first-order sentences would be interpreted in potentially infinite domains.

Definition (Satisfaction in sufficiently large models)

For any formula $\phi(x_1, \ldots, x_r)$ and $a_1, \ldots, a_r \in \omega$, we say that $\phi$ is sl-satisfied in $FM(A)$ by $a_1, \ldots, a_r$ ($FM(A) \models_{sl} \phi(a_1, \ldots, a_r)$) if and only if $\exists m \forall k (k \geq m \Rightarrow A_k \models \phi(a_1, \ldots, a_r))$.
The notion of algorithmic learnability is one of the equivalent formulations of the concept of methods proceeding by mind-changes and stabilizing on every input. One of them is the notion of $\text{FM}$-representability proposed by M. Mostowski.
The notion of algorithmic learnability is one of the equivalent formulations of the concept of methods proceeding by mind-changes and stabilizing on every input. One of them is the notion of FM-representability proposed by M. Mostowski.

Primary motivation: computational foundations of mathematics and the search for the semantics under which first-order sentences would be interpreted in potentially infinite domains.
SL-semantics

The notion of algorithmic learnability is one of the equivalent formulations of the concept of methods proceeding by mind-changes and stabilizing on every input. One of them is the notion of FM-representability proposed by M. Mostowski. Primary motivation: computational foundations of mathematics and the search for the semantics under which first-order sentences would be interpreted in potentially infinite domains.

**Definition (Satisfaction in sufficiently large models)**

For any formula \( \varphi(x_1, \ldots, x_r) \) and \( a_1, \ldots, a_r \in \omega \) we say that \( \varphi \) is sl-satisfied in \( \text{FM}(A) \) by \( a_1, \ldots, a_r \) (\( \text{FM}(A) \models_{\text{sl}} \varphi(a_1, \ldots, a_r) \)) if and only if

\[
\exists m \forall k \ (k \geq m \ \Rightarrow \ A_k \models \varphi(a_1, \ldots, a_r)).
\]
More generally we could say that for a given class $\mathcal{K}$ of finite models

$$sl(\mathcal{K}) = \{ \varphi \in \text{Sent}_{L_\sigma} : \exists n \forall \mathcal{M} \in \mathcal{K} \ (\text{card}(\mathcal{M}) \geq n \Rightarrow \mathcal{M} \models \varphi) \}.$$
More generally we could say that for a given class $\mathcal{K}$ of finite models

$$sl(\mathcal{K}) = \{ \varphi \in \text{Sent}_{\mathcal{L}_\sigma} : \exists n \forall \mathcal{M} \in \mathcal{K} \ (\text{card}(\mathcal{M}) \geq n \Rightarrow \mathcal{M} \models \varphi) \}.$$ 

**Definition**

A set of sentences $\Delta$ sl-entails a formula $\varphi$, denoted by: $\Delta \models_{sl} \varphi$ if and only if for any given class $\mathcal{K}$ of finite models we have:

$$\text{if } \mathcal{K} \models_{sl} \Delta, \text{ then } \mathcal{K} \models_{sl} \varphi.$$
Logic of (sufficiently large) finite models

When a vocabulary $\sigma$ is fixed then we define sl-logic as $L_{sl} := sl(\text{MOD}_\sigma)$.

The logic of finite models is defined as $L_{fin} := th(\text{MOD}_\sigma)$.

Let $A_{inf}$ denote the family of sentences $\{\varphi_n : n \geq 2\}$, where $\varphi_n$ is the following:

$$\exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j.$$
Logic of (sufficiently large) finite models

When a vocabulary $\sigma$ is fixed then we define sl-logic as $L_{sl} := sI(\text{MOD}_\sigma)$.

The logic of finite models is defined as $L_{\text{fin}} := th(\text{MOD}_\sigma)$.

Let $A_{inf}$ denote the family of sentences $\{\varphi_n : n \geq 2\}$, where $\varphi_n$ is the following:

$$\exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j.$$ 

Let $A_{sl} = L_{\text{fin}} \cup A_{inf}$. Then we have the following (all due to M. Mostowski):

1. $L_{sl}$ is the set of all first-order consequences of $A_{sl}$.
2. $A_{sl}$ is $\Pi^0_1$-complete.
3. $L_{sl}$ is $\Sigma^0_2$-complete.
4. Completeness: For any set of $\sigma$-sentences $T$ containing $A_{sl}$ and any $\sigma$-sentence $\varphi$ we have:

$$T \vdash \varphi \iff T \models_{sl} \varphi.$$
Logic of (sufficiently large) finite models

When a vocabulary $\sigma$ is fixed then we define sl-logic as $L_{sl} := sl(\text{MOD}_\sigma)$.

The logic of finite models is defined as $L_{fin} := th(\text{MOD}_\sigma)$.

Let $A_{inf}$ denote the family of sentences $\{\varphi_n : n \geq 2\}$, where $\varphi_n$ is the following:

$$\exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j.$$

Let $A_{sl} = L_{fin} \cup A_{inf}$. Then we have the following (all due to M. Mostowski):

**Theorem**

1. $L_{sl}$ is the set of all first-order consequences of $A_{sl}$.
2. $A_{sl}$ is $\Pi^0_1$-complete.
3. $L_{sl}$ is $\Sigma^0_2$-complete.
4. Completeness: For any set of $\sigma$-sentences $T$ containing $A_{sl}$ and any $\sigma$-sentence $\varphi$ we have:

$$T \vdash \varphi \iff T \models_{sl} \varphi.$$
FM-representability

We say that the relation $R \subseteq \omega^r$ is FM-represented in $FM(A)$ by a formula $\varphi(x_1, \ldots, x_r)$ if and only if for each $a_1, \ldots, a_r \in \omega$ both of the following conditions hold:

1. $R(a_1, \ldots, a_r)$ if and only if $\text{FM}(A)|= sl\varphi(a_1, \ldots, a_r)$.
2. $\neg R(a_1, \ldots, a_r)$ if and only if $\text{FM}(A)|= sl\neg \varphi(a_1, \ldots, a_r)$.

We say that $R$ is FM-representable in $FM(A)$ if there is a formula $\varphi$ such that it FM-represents $R$ in $FM(A)$.

If a relation is FM-representable in $FM(N)$ we say that it is FM-representable.
FM-representability - a model of the semantic meaningfulness of mathematical concepts that we learn:

Definition (FM-representability)

We say that the relation $R \subseteq \omega^r$ is FM-represented in $\mathsf{FM}(A)$ by a formula $\varphi(x_1, \ldots, x_r)$ if and only if for each $a_1, \ldots, a_r \in \omega$ both of the following conditions hold:

1. $R(a_1, \ldots, a_r)$ if and only if $\mathsf{FM}(A) \models \varphi(a_1, \ldots, a_r)$.
2. $\neg R(a_1, \ldots, a_r)$ if and only if $\mathsf{FM}(A) \models \neg \varphi(a_1, \ldots, a_r)$.

We say that $R$ is FM-representable in $\mathsf{FM}(A)$ if there is a formula $\varphi$ such that it FM-represents $R$ in $\mathsf{FM}(A)$.

If a relation is FM-representable in $\mathsf{FM}(\mathbb{N})$ we say that it is FM-representable.
FM-representability - a model of the semantic meaningfulness of mathematical concepts that we learn:

**Definition (FM-representability)**

We say that the relation $R \subseteq \omega^r$ is FM-represented in $FM(A)$ by a formula $\varphi(x_1, \ldots, x_r)$ if and only if for each $a_1, \ldots, a_r \in \omega$ both of the following conditions hold:

1. $R(a_1, \ldots, a_r)$ if and only if $FM(A) \models_{sl} \varphi(a_1, \ldots, a_r)$.
2. $\neg R(a_1, \ldots, a_r)$ if and only if $FM(A) \models_{sl} \neg \varphi(a_1, \ldots, a_r)$.
FM-representability - a model of the semantic meaningfulness of mathematical concepts that we learn:

Definition (FM-representability)

We say that the relation \( R \subseteq \omega^r \) is FM-represented in \( \text{FM}(A) \) by a formula \( \varphi(x_1, \ldots, x_r) \) if and only if for each \( a_1, \ldots, a_r \in \omega \) both of the following conditions hold:

1. \( R(a_1, \ldots, a_r) \) if and only if \( \text{FM}(A) \models_{sl} \varphi(a_1, \ldots, a_r) \).
2. \( \neg R(a_1, \ldots, a_r) \) if and only if \( \text{FM}(A) \models_{sl} \neg \varphi(a_1, \ldots, a_r) \).

We say that \( R \) is **FM-representable in** \( \text{FM}(A) \) if there is a formula \( \varphi \) such that it FM-represents \( R \) in \( \text{FM}(A) \).
FM-representability

FM-representability - a model of the semantic meaningfulness of mathematical concepts that we learn:

**Definition (FM-representability)**

We say that the relation \( R \subseteq \omega^r \) is FM-represented in \( \text{FM}(A) \) by a formula \( \varphi(x_1, \ldots, x_r) \) if and only if for each \( a_1, \ldots, a_r \in \omega \) both of the following conditions hold:

\[
\begin{align*}
1 & \quad R(a_1, \ldots, a_r) \text{ if and only if } \text{FM}(A) \models_{sl} \varphi(a_1, \ldots, a_r). \\
2 & \quad \neg R(a_1, \ldots, a_r) \text{ if and only if } \text{FM}(A) \models_{sl} \neg \varphi(a_1, \ldots, a_r).
\end{align*}
\]

We say that \( R \) is **FM-representable** in \( \text{FM}(A) \) if there is a formula \( \varphi \) such that it FM-represents \( R \) in \( \text{FM}(A) \).

If a relation is FM-representable in \( \text{FM}(\mathbb{N}) \) we say that it is **FM-representable**.
Limit Lemma

There is a collection of notions that turned out to be equivalent to algorithmic learnability.
There is a collection of notions that turned out to be equivalent to algorithmic learnability.

**Theorem (Limit Lemma)**

Let $R \subseteq \omega$. Then the following are equivalent:

1. $R$ is recursive with recursively enumerable oracle,
2. $\deg(R) \leq 0'$,
3. $R$ is algorithmically learnable,
4. $R$ is $\Delta^0_2$,
5. $R$ is FM-representable.


Czarnecki, Godziszewski, Kalociński

Learnability and Church’s Thesis

CUNY LW 2016
There is a collection of notions that turned out to be equivalent to algorithmic learnability.

**Theorem (Limit Lemma)**

Let $R \subseteq \omega$. Then the following are equivalent:

1. $R$ is recursive with recursively enumerable oracle,
2. $\deg(R) \leq 0'$,
3. $R$ is algorithmically learnable,
4. $R$ is $\Delta^0_2$,
5. $R$ is FM-representable.
There is a collection of notions that turned out to be equivalent to algorithmic learnability.

**Theorem (Limit Lemma)***

Let \( R \subseteq \omega \). Then the following are equivalent:

1. \( R \) is recursive with recursively enumerable oracle,
2. \( \deg(R) \leq 0' \),
3. \( R \) is algorithmically learnable,
4. \( R \) is \( \Delta^0_2 \),
5. \( R \) is FM-representable.
There is a collection of notions that turned out to be equivalent to algorithmic learnability.

**Theorem (Limit Lemma)**

Let $R \subseteq \omega$. Then the following are equivalent:

1. $R$ is recursive with recursively enumerable oracle,
2. $\text{deg}(R) \leq 0'$,
3. $R$ is algorithmically learnable,
Limit Lemma

There is a collection of notions that turned out to be equivalent to algorithmic learnability.

**Theorem (Limit Lemma)**

Let $R \subseteq \omega$. Then the following are equivalent:

1. $R$ is recursive with recursively enumerable oracle,
2. $\deg(R) \leq 0'$,
3. $R$ is algorithmically learnable,
4. $R$ is $\Delta^0_2$. 

Czarnecki, Godziszewski, Kalociński
Learnability and Church’s Thesis

CUNY LW 2016
Limit Lemma

There is a collection of notions that turned out to be equivalent to algorithmic learnability.

**Theorem (Limit Lemma)**

Let $R \subseteq \omega$. Then the following are equivalent:

1. $R$ is recursive with recursively enumerable oracle,
2. $\text{deg}(R) \leq 0'$,
3. $R$ is algorithmically learnable,
4. $R$ is $\Delta^0_2$,
5. $R$ is FM-representable.
Interlude: FM-definability and hierarchies of functions

Definition
R is FM-definable via f if there is an arithmetical formula $\varphi(x)$ such that for any $n \in \omega$, $\varphi(x)$ defines $R \upharpoonright n$ in $\mathbb{N}^m$, for any $m \geq f(n)$.

Definition
The class of functions FM-definable via $f$ is denoted by $\text{FM}(f)$. Let $X$ be the class of partial functions. Let $\text{Tot}(X)$ denote the maximal class $Y \subseteq X$ consisting of total functions. We define $\text{FM}(X) = \bigcup_{f \in \text{Tot}(X)} \text{FM}(f)$.

Proposition
For any relation $R$ the following are equivalent:
there exists a function $f$ such that $R$ is FM-definable via $f$
there exists a function $g \leq 0'$ such that $R$ is FM-definable via $g$. 
Interlude: FM-definability and hierarchies of functions

Definition

$R$ is FM-definable via $f$ if there is an arithmetical formula $\varphi(x)$ such that for any $n \in \omega$, $\varphi(x)$ defines $R \upharpoonright n$ in $\mathbb{N}_m$, for any $m \geq f(n)$. 

Definition

The class of functions FM-definable via $f$ is denoted by $\text{FM}(f)$. Let $X$ be the class of partial functions. Let $\text{Tot}(X)$ denote the maximal class $Y \subseteq X$ consisting of total functions. We define $\text{FM}(X) = \bigcup_{f \in \text{Tot}(X)} \text{FM}(f)$. 

Proposition

For any relation $R$ the following are equivalent:

1. There exists a function $f$ such that $R$ is FM-definable via $f$.
2. There exists a function $g \leq 0'$ such that $R$ is FM-definable via $g$. 

Czarnecki, Godziszewski, Kalociński
Learnability and Church’s Thesis
CUNY LW 2016
Interlude: FM-definability and hierarchies of functions

**Definition**

R is FM-definable via f if there is an arithmetical formula \( \varphi(x) \) such that for any \( n \in \omega \), \( \varphi(x) \) defines \( R \upharpoonright n \) in \( \mathbb{N}_m \), for any \( m \geq f(n) \).

**Definition**

The class of functions FM-definable via f is denoted by \( FM(f) \). Let \( X \) be the class of partial functions. Let \( Tot(X) \) denote the maximal class \( Y \subseteq X \) consisting of total functions. We define \( FM(X) = \bigcup_{f \in Tot(X)} FM(f) \).
Interlude: FM-definability and hierarchies of functions

**Definition**

R is FM-definable via f if there is an arithmetical formula $\varphi(x)$ such that for any $n \in \omega$, $\varphi(x)$ defines $R \upharpoonright n$ in $\mathbb{N}_m$, for any $m \geq f(n)$.

**Definition**

The class of functions FM-definable via f is denoted by $\text{FM}(f)$. Let $X$ be the class of partial functions. Let $\text{Tot}(X)$ denote the maximal class $Y \subseteq X$ consisting of total functions. We define $\text{FM}(X) = \bigcup_{f \in \text{Tot}(X)} \text{FM}(f)$.

**Proposition**

For any relation $R$ the following are equivalent:

- there exists a function $f$ such that $R$ is FM-definable via $f$
- there exists a function $g \leq 0'$ such that $R$ is FM-definable via $g$. 
Interlude: FM-definability and hierarchies of functions

**Definition**

*f is majorized by g* if \( \exists k \forall n > k \ f(n) < g(n) \).

**Question**

Let \( a, b \) be Turing degrees. Does the following equivalence hold:

\[ a <_T b \iff a <_M b \]
Interlude: FM-definability and hierarchies of functions

Definition

\( f \) is majorized by \( g \) if \( \exists k \forall n > k \ f(n) < g(n) \).

Definition

Let \( X, Y \) be classes of total functions. Say \( X \leq_M Y \) if any \( f \in X \) is majorized by some \( g \in Y \).
Interlude: FM-definability and hierarchies of functions

**Definition**

**f** is majorized by **g** if \( \exists k \forall n > k \ f(n) < g(n) \).

**Definition**

Let \( X, Y \) be classes of total functions. Say \( X \leq_M Y \) if any \( f \in X \) is majorized by some \( g \in Y \).

**Question**

Let \( a, b \) be Turing degrees. Does the following equivalence hold:

\[
a <_T b \iff a <_M b
\]
Interlude: FM-definability and hierarchies of functions

Question

Let $a, b$ be Turing degrees. Does the following equivalence hold:

$$a <_T b \iff a <_M b?$$
Interlude: FM-definability and hierarchies of functions

Question

Let \( a, b \) be Turing degrees. Does the following equivalence hold:

\[
\text{a} <_T \text{b} \iff \text{a} <_M \text{b}?
\]

Definition

Let \( F, G \subseteq \omega^\omega \). We write \( F \simeq G \) if \( F \leq_M G \) and \( G \leq_M F \).
Interlude: FM-definability and hierarchies of functions

**Question**

Let $a, b$ be Turing degrees. Does the following equivalence hold:

$$a <_T b \iff a <_M b?$$

**Definition**

Let $F, G \subseteq \omega^\omega$. We write $F \simeq G$ if $F \leq_M G$ and $G \leq_M F$.

**Lemma**

Let $F, G$ be classes of total functions such that $F \simeq G$. Then $FM(F) = FM(G)$.
Interlude: FM-definability and hierarchies of functions

Question

Let \( a, b \) be Turing degrees. Does the following equivalence hold:

\[ a \lt_T b \iff a \lt_M b? \]

Definition

Let \( F, G \subseteq \omega^\omega \). We write \( F \simeq G \) if \( F \leq_M G \) and \( G \leq_M F \).

Lemma

Let \( F, G \) be classes of total functions such that \( F \simeq G \). Then \( FM(F) = FM(G) \).

Lemma

Let \( a, b \) be Turing degrees. If \( a \leq_T b \) then \( a \leq_M b \).
Interlude: FM-definability and hierarchies of functions

Theorem

\( a <_T b \Rightarrow a <_M b. \)

Proof. Assume \( a <_T b \). For the sake of contradiction, assume \( a \not< M b \) which means that \( a \not\leq M b \) or \( b \leq M a \). However, by Lemma 2, \( a <_T b \) implies \( a \leq M b \). So we have \( b \leq M a \). But then, by Lemma 1, we get \( FM(Tot(a)) = FM(Tot(b)) \). This gives \( a = b \).
Interlude: FM-definability and hierarchies of functions

Theorem

\[
\text{a } <_T \text{ b } \Rightarrow \text{ a } <_M \text{ b.}
\]

Proof.

Assume \( a <_T b \). For the sake of contradiction, assume \( a \not<_M b \) which means that \( a \not<_M b \) or \( b \leq_M a \). However, by Lemma 2, \( a <_T b \) implies \( a \leq_M b \). So we have \( b \leq_M a \). But then, by Lemma 1, we get

\[
FM(Tot(a)) = FM(Tot(b)).
\]

This gives \( a = b \).
Interlude: FM-definability and hierarchies of functions

Theorem
\( a <_T b \Rightarrow a <_M b. \)

Proof.
Assume \( a <_T b. \) For the sake of contradiction, assume \( a \not<_M b \) which means that \( a \not<_M b \) or \( b \leq_M a. \) However, by Lemma 2, \( a <_T b \) implies \( a \leq_M b. \) So we have \( b \leq_M a. \) But then, by Lemma 1, we get \( FM(Tot(a)) = FM(Tot(b)). \) This gives \( a = b. \)

Question
\( a <_M b \Rightarrow a <_T b? \)
Interlude: Witnessing functions

Definition (Witnessing function)

Let $g$ be a total function and $A_s$ a recursive approximation of $A = \lim_{s \in \Delta_0^2} A_s$. We say $g$ is a witnessing function for $A$ wrt $A_s$ if for every $x \in \mathbb{N}$ we have:

$$x \in A \iff \forall s \geq g(x) \ A_s(x) = A(x)$$

We say $g$ is a minimal witnessing function for $A$ wrt $A_s$, if for every witnessing function $g'$ for $A$ wrt $A_s$ we have:

$$\forall x \ g(x) \leq g'(x)$$

We say $g$ is a (minimal) witnessing function if there is some $A$ and a recursive approximation $A_s$ such that $g$ is a (minimal) witnessing function for $A$ wrt $A_s$. 
Interlude: Witnessing functions

Definition (Witnessing function)

Let $g$ be a total function and $A_s$ a recursive approximation of

$$A = \lim_s A_s \in \Delta^0_2.$$  

We say $g$ is a **witnessing function** for $A$ wrt $A_s$ if for every $x \in \mathbb{N}$ we have

$$x \in A \iff \forall s \geq g(x) \ A_s(x) = A(x).$$

We say $g$ is a **minimal witnessing function** if for every witnessing function $g'$ for $A$ wrt. $A_s$ we have

$$\forall x \ g(x) \leq g'(x).$$

We say $g$ is a **(minimal) witnessing function** if there is some $A$ and a recursive approximation $A_s$ such that $g$ is a (minimal) witnessing function for $A$ wrt. $A_s$.  

Czarnecki, Godziszewski, Kalociński
Learnability and Church’s Thesis
CUNY LW 2016
16 / 41
Definition (Witnessing function)

Let \( g \) be a total function and \( A_s \) a recursive approximation of \( A = \lim_s A_s \in \Delta_2^0 \). We say \( g \) is a \textbf{witnessing function for} \( A \) \textbf{wrt} \( A_s \) if for every \( x \in \mathbb{N} \) we have

\[
x \in A \iff \forall s \geq g(x) A_s(x) = A(x)
\]

We say \( g \) is a \textbf{minimal witnessing function for} \( A \) \textbf{wrt} \( A_s \), if for every witnessing function \( g' \) for \( A \) \textbf{wrt}. \( A_s \) we have \( \forall x \ g(x) \leq g'(x) \).
Interlude: Witnessing functions

**Definition (Witnessing function)**

Let $g$ be a total function and $A_s$ a recursive approximation of $A = \lim_s A_s \in \Delta^0_2$. We say $g$ is a **witnessing function for** $A$ wrt $A_s$ if for every $x \in \mathbb{N}$ we have

$$x \in A \iff \forall s \geq g(x) A_s(x) = A(x)$$

We say $g$ is a **minimal witnessing function for** $A$ wrt $A_s$, if for every witnessing function $g'$ for $A$ wrt. $A_s$ we have $\forall x g(x) \leq g'(x)$.

We say $g$ is a **(minimal) witnessing function** if there is some $A$ and a recursive approximation $A_s$ such that $g$ is a (minimal) witnessing function for $A$ wrt. $A_s$. 

Interlude: Witnessing functions

Observe that if $g$ is a minimal witnessing function for $A = \lim_s A_s$ and $g'$ is any witnessing function for it, then $g \leq_T g'$.
Interlude: Witnessing functions

Observe that if \( g \) is a minimal witnessing function for \( A = \lim_s A_s \) and \( g' \) is any witnessing function for it, then \( g \leq_T g' \).

We may ask what are degrees of minimal witnessing functions, e.g. is every set \( \leq_T 0' \) Turing-equivalent to its minimal witnessing function?
Interlude: Witnessing functions

Observe that if $g$ is a minimal witnessing function for $A = \lim_s A_s$ and $g'$ is any witnessing function for it, then $g \leq_T g'$. We may ask what are degrees of minimal witnessing functions, e.g. is every set $\leq_T 0'$ Turing-equivalent to its minimal witnessing function?

**Theorem**

*Every minimal witnessing function is of r.e. degree.*

Hence, for $A \in \Delta^0_2$ which is not of r.e. degree and for its minimal witnessing function $g$, we cannot have $\text{deg}(A) = \text{deg}(g)$. 
Interlude: Witnessing functions

Observe that if $g$ is a minimal witnessing function for $A = \lim_s A_s$ and $g'$ is any witnessing function for it, then $g \leq_T g'$.

We may ask what are degrees of minimal witnessing functions, e.g. is every set $\leq_T 0'$ Turing-equivalent to its minimal witnessing function?

**Theorem**

*Every minimal witnessing function is of r.e. degree.*

Hence, for $A \in \Delta^0_2$ which is not of r.e. degree and for its minimal witnessing function $g$, we cannot have $\deg(A) = \deg(g)$.

However:

**Theorem**

*Let $A \in \Delta^0_2$. If $A$ is weakly 1-r.e. then $A$ has a minimal witnessing function $g$ such that $\deg(A) = \deg(g)$.*
Interlude: Witnessing functions

What is more:
What is more:

**Proposition**

Every r.e. degree contains a minimal witnessing function.
Interlude: Witnessing functions

What is more:

**Proposition**
Every r.e. degree contains a minimal witnessing function.

**Theorem**
There exist Turing-incomparable functions \( f, g < 0' \) witnessing the same non-recursive set.

(in fact: let \( f \) be any non-recursive function \( \leq 0' \), whose values are recursively bounded. Then \( f \) does not witness any set \( A \in \Delta^0_2 - 0 \).)
Interlude: Witnessing functions

What is more:

**Proposition**
Every r.e. degree contains a minimal witnessing function.

**Theorem**
There exist Turing-incomparable functions $f, g < 0'$ witnessing the same non-recursive set.

**Proposition**
There exists a non-recursive total function $\leq 0'$ which does not witness any set $A \in \Delta^0_2 - 0$.

(in fact: let $f$ be any non-recursive function $\leq 0'$, whose values are recursively bounded. Then $f$ does not witness any set $A \in \Delta^0_2 - 0$.)
Interlude: Witnessing functions

But there still are questions left:
Interlude: Witnessing functions

But there still are questions left:

**Question**

Is every total function $\leq 0'$, that is not recursively bounded, a witnessing function for some non-recursive set?
But there still are questions left:

**Question**

*Is every total function \( \leq 0' \), that is not recursively bounded, a witnessing function for some non-recursive set?*

**Question**

*Let \( a < b \leq 0' \). Does every \( A \in a \) has a witnessing function in \( b \)?*
Interlude: Witnessing functions

But there still are questions left:

**Question**

Is every total function $\leq 0'$, that is not recursively bounded, a witnessing function for some non-recursive set?

**Question**

Let $a < b \leq 0'$. Does every $A \in a$ has a witnessing function in $b$?

**Question**

Let $a < b \leq 0'$ and assume $b$ is r.e. Does every $A \in a$ has a minimal witnessing function in $b$?
Learnability Thesis

Back to philosophy:
Back to philosophy:
We face the question of whether the notion of intuitive learnability is equivalent to the notion of algorithmic learnability.
Learnability Thesis

Back to philosophy:
We face the question of whether the notion of intuitive learnability is equivalent to the notion of algorithmic learnability.
In other words, is the class of intuitively learnable sets equal to the class of algorithmically learnable sets?
Learnability Thesis

Back to philosophy:
We face the question of whether the notion of intuitive learnability is equivalent to the notion of algorithmic learnability. In other words, is the class of intuitively learnable sets equal to the class of algorithmically learnable sets?

- $\mathcal{IL}$ - the class of all intuitively learnable sets of natural numbers.
Learnability Thesis

Back to philosophy:
We face the question of whether the notion of intuitive learnability is equivalent to the notion of algorithmic learnability. In other words, is the class of intuitively learnable sets equal to the class of algorithmically learnable sets?

- $\mathcal{IL}$ - the class of all intuitively learnable sets of natural numbers.
- $\Delta^0_2$ - the class of algorithmically learnable sets.
Learnability Thesis

Back to philosophy:
We face the question of whether the notion of intuitive learnability is equivalent to the notion of algorithmic learnability.
In other words, is the class of intuitively learnable sets equal to the class of algorithmically learnable sets?

- $\mathcal{IL}$ - the class of all intuitively learnable sets of natural numbers.
- $\Delta^0_2$ - the class of algorithmically learnable sets.

The Learnability Thesis presents shortly as follows:
Learnability Thesis

Back to philosophy:
We face the question of whether the notion of intuitive learnability is equivalent to the notion of algorithmic learnability. In other words, is the class of intuitively learnable sets equal to the class of algorithmically learnable sets?

- $\mathcal{IL}$ - the class of all intuitively learnable sets of natural numbers.
- $\Delta_2^0$ - the class of algorithmically learnable sets.

The Learnability Thesis presents shortly as follows:

**Thesis (Learnability Thesis)**

$\mathcal{IL} = \Delta_2^0$. 
Proposition

The Church’s Thesis entails the Learnability Thesis.

Proof.

Assume the Church’s Thesis. Let \( A \in \Delta^0_2 \subseteq IL \). Let \( g : \mathbb{N}^2 \rightarrow \{0, 1\} \) be algorithmically learnable. By the Church’s Thesis, \( g \) is an intuitively computable total function. Devise an intuitively learnable infinite procedure for \( A \): let \( x \in \mathbb{N} \). Set \( t = 0 \). In infinite loop do: intuitively compute \( g(t, x) \), output the result in case it differs from the result obtained previously, increment \( t \). This shows \( A \in IL \).
Proposition

The Church’s Thesis entails the Learnability Thesis.

Proof.

Assume the Church’s Thesis. \((\Delta_2^0 \subseteq \mathcal{IL})\) Let \(A \in \Delta_2^0\). Let \(g : \omega^2 \rightarrow \{0, 1\}\) be algorithmically learnable. By the Church’s Thesis, \(g\) is an intuitively computable total function. Devise an intuitively learnable infinite procedure for \(A\): let \(x \in \omega\). Set \(t = 0\). In infinite loop do: intuitively compute \(g(t, x)\), output the result in case it differs from the result obtained previously, increment \(t\). This shows \(A \in \mathcal{IL}\).
Proposition

The Church’s Thesis entails the Learnability Thesis.

Proof.

Assume the Church’s Thesis. ($\mathcal{IL} \subseteq \Delta^0_2$) Let $A \in \mathcal{IL}$. Then there is an intuitive algorithm, say $G$, learning $A$. Devise an intuitive algorithm $G'$ that takes $(t, x)$ as input and returns the last answer generated by $G$ on input $x$ up to $t$ steps of intuitive computation. By the Church’s Thesis, the function intuitively computed by $G'$ is recursive. Let $g(t, x)$ be that function. Clearly, $g$ is total and satisfies the definition of algorithmic learnability. Hence, by the Limit Lemma, $A$ is $\Delta^0_2$. 

Question: what about the other direction?
Proposition

The Church’s Thesis entails the Learnability Thesis.

Proof.

Assume the Church’s Thesis. \((\mathcal{IL} \subseteq \Delta^0_2)\) Let \(A \in \mathcal{IL}\). Then there is an intuitive algorithm, say \(G\), learning \(A\). Devise an intuitive algorithm \(G'\) that takes \((t, x)\) as input and returns the last answer generated by \(G\) on input \(x\) up to \(t\) steps of intuitive computation. By the Church’s Thesis, the function intuitively computed by \(G'\) is recursive. Let \(g(t, x)\) be that function. Clearly, \(g\) is total and satisfies the definition of algorithmic learnability. Hence, by the Limit Lemma, \(A\) is \(\Delta^0_2\).

Question: what about the other direction?
Testing formulae

Let $R \subseteq \omega^n$ and $\phi(x_1, \ldots, x_n)$ be a formula. A formula $\psi(x_1, \ldots, x_n)$ is a testing formula for $\phi(x_1, \ldots, x_n)$ and $R$ if:

1. For each $a_1, \ldots, a_n \in \omega$, there is $n_0 \in \omega$ such that for each finite model $M$, $M|\psi(a_1, \ldots, a_n)$ if and only if $|M| \geq n_0$.
2. For each $a_1, \ldots, a_n \in \omega$ and each finite model $M$, if $M|\psi(a_1, \ldots, a_n)$, then $R(a_1, \ldots, a_n)$ if and only if $M|\phi(a_1, \ldots, a_n)$.

The conditions defining the notion of testing formula for $\phi$ and $R$ may be read as an explication of the concept of knowing the answer (and achieving the answer effectively) to the query of the form: is a tuple $a_1, \ldots, a_n$ in the relation $R$?
Definition (Testing formula)

Let $R \subseteq \omega^n$ and $\varphi(x_1, \ldots, x_n)$ be a formula. A formula $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and $R$ if:

1. For each $a_1, \ldots, a_n \in \omega$ there is $n_0 \in \omega$ such that for each finite model $M$, $M|\psi(a_1, \ldots, a_n)$ if and only if $|M| \geq n_0$.
2. For each $a_1, \ldots, a_n \in \omega$ and each finite model $M$, if $M|\psi(a_1, \ldots, a_n)$, then $R(a_1, \ldots, a_n)$ if and only if $M|\varphi(a_1, \ldots, a_n)$.

The conditions defining the notion of testing formula for $\varphi$ and $R$ may be read as an explication of the concept of knowing the answer (and achieving the answer effectively) to the query of the form: is a tuple $a_1, \ldots, a_n$ in the relation $R$?
Testing formulae

Definition (Testing formula)

Let $R \subseteq \omega^n$ and $\varphi(x_1, \ldots, x_n)$ be a formula. A formula $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and $R$ if:

- for each $a_1, \ldots, a_n \in \omega$ there is $n_0 \in \omega$ such that for each finite model $M$,
  $M \models \psi(a_1, \ldots, a_n)$ if and only if $|M| \geq n_0$.
Definition (Testing formula)

Let $R \subseteq \omega^n$ and $\varphi(x_1, \ldots, x_n)$ be a formula. A formula $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and $R$ if:

- for each $a_1, \ldots, a_n \in \omega$ there is $n_0 \in \omega$ such that for each finite model $M$, $M \models \psi(a_1, \ldots, a_n)$ if and only if $|M| \geq n_0$,

- for each $a_1, \ldots, a_n \in \omega$ and each finite model $M$, if $M \models \psi(a_1, \ldots, a_n)$, then $R(a_1, \ldots, a_n)$ if and only if $M \models \varphi(a_1, \ldots, a_n)$.
Testing formulae

Definition (Testing formula)

Let $R \subseteq \omega^n$ and $\varphi(x_1, \ldots, x_n)$ be a formula. A formula $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and $R$ if:

- for each $a_1, \ldots, a_n \in \omega$ there is $n_0 \in \omega$ such that for each finite model $M$, $M \models \psi(a_1, \ldots, a_n)$ if and only if $|M| \geq n_0$,

- for each $a_1, \ldots, a_n \in \omega$ and each finite model $M$, if $M \models \psi(a_1, \ldots, a_n)$, then $R(a_1, \ldots, a_n)$ if and only if $M \models \varphi(a_1, \ldots, a_n)$.

The conditions defining the notion of testing formula for $\varphi$ and $R$ may be read as an explication of the concept of knowing the answer (and achieving the answer effectively) to the query of the form: is a tuple $a_1, \ldots, a_n$ in the relation $R$?
Testing formulae then serve the epistemological criterion of separating decidable relations from other FM-representable notions:

\[
\text{Theorem (Mostowski)}
\]

Let \( R \subseteq \omega^n \). \( R \) is decidable if and only if there are formulae \( \varphi(x_1, \ldots, x_n) \), \( \psi(x_1, \ldots, x_n) \) such that \( \psi(x_1, \ldots, x_n) \) is a testing formula for \( \varphi(x_1, \ldots, x_n) \) and \( R \).
Testing formulae then serve the epistemological criterion of separating decidable relations from other FM-representable notions:

**Theorem (Mostowski)**

Let \( R \subseteq \omega^n \). \( R \) is decidable if and only if there are formulae \( \varphi(x_1, \ldots, x_n) \), \( \psi(x_1, \ldots, x_n) \) such that \( \psi(x_1, \ldots, x_n) \) is a testing formula for \( \varphi(x_1, \ldots, x_n) \) and \( R \).
Theorem (M. Mostowski)

Let \( R \subseteq \omega^n \). \( R \) is decidable if and only if there are formulae \( \varphi(x_1, \ldots, x_n) \), \( \psi(x_1, \ldots, x_n) \) such that \( \psi(x_1, \ldots, x_n) \) is a testing formula for \( \varphi(x_1, \ldots, x_n) \) and \( R \).

Proof

Fix \( R \subseteq \omega^n \). (\( \Rightarrow \)) Let \( T(e, x_1, \ldots, x_n, c) \) be the Kleene predicate: \( c \) is the computation of the algorithm \( e \) on input \( x_1, \ldots, x_n \). Let \( U(c, y) \) mean that a computation with code \( c \) accepts if \( y = 1 \) or rejects if \( y = 0 \). Suppose that \( R \) is decidable by an algorithm with the code \( e \). We define:

\[
\psi(x_1, \ldots, x_n) = \exists c \ T(e, x_1, \ldots, x_n, c),
\]

\[
\varphi(x_1, \ldots, x_n) = \exists c \ (T(e, x_1, \ldots, x_n, c) \land U(c, 1)).
\]
Proof ...

Fix $\bar{a} = a_1, \ldots, a_n \in \omega$. We show that $\psi$ is a testing formula for $\varphi$ and $R$.

We have $\mathbb{N} \models \exists c \ T(e, \bar{a}, c)$ thus for some $n_0 \in \omega$ it holds that $\mathbb{N} \models T(e, \bar{a}, n_0)$. Since the computation of $e$ on $\bar{a}$ is unique, so is $n_0$. Therefore for $m \in \omega$, $\mathbb{N}_m \models \psi(\bar{a})$ if and only if $m \geq n_0$.

Now fix $m \in \omega$ such that $\mathbb{N}_m \models \psi(\bar{a})$. Let $n_0 \in \omega$ be such that $\mathbb{N} \models T(e, \bar{a}, n_0)$. Then for every $m \geq n_0$ it holds that $\mathbb{N}_m \models T(e, \bar{a}, n_0)$. If $R(\bar{a})$, then $\mathbb{N} \models U(n_0, 1)$ and $\mathbb{N}_m \models \varphi(\bar{a})$. On the other hand if $\neg R(\bar{a})$, then $\mathbb{N} \models U(n_0, 0)$ and $\mathbb{N}_m \models \neg \varphi(\bar{a})$.

Therefore $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi$ and $R$. 
Proof......

(⇐) Let \( \psi(x_1, \ldots, x_n) \) be a testing formula for \( \varphi(x_1, \ldots, x_n) \) and \( R \). The algorithm deciding \( R \) is the following.

**input**: \( a_1, \ldots, a_n \in \omega \)

**output**: truth value of \( R(a_1, \ldots, a_n) \)

\[
\begin{align*}
i &:= 0 \\
\text{while } &\not\models \psi(a_1, \ldots, a_n) \\
&\quad i := i + 1
\end{align*}
\]

**return**: truth value of \( \models \varphi(a_1, \ldots, a_n) \)

The algorithm implicitly uses subroutines to compute \( i \mapsto \lceil \mathbb{N}_i \rceil \) and \( \mathbb{N}_i \models \alpha \) which are both recursive. It also always halts since \( \psi(x_1, \ldots, x_n) \) is a testing formula for \( \varphi(x_1, \ldots, x_n) \) and \( R \). This ends the proof.
Mostowski proved that using testing formulae enables us to prove the implication: $\mathcal{IL} = \Delta^0_2 \Rightarrow \mathcal{IC} = \Delta^0_1$, if taken together with certain additional assumptions.
Mostowski proved that using testing formulae enables us to prove the implication: $\mathcal{IL} = \Delta^0_2 \Rightarrow \mathcal{IC} = \Delta^0_1$, if taken together with certain additional assumptions.

**Mostowski’s assumptions**

1. There is a recursive enumeration of finite models,
2. Every finite model $\mathbb{N}_m$ has a recursive satisfaction relation.
Mostowski proved that using testing formulae enables us to prove the implication: $\mathcal{I}L = \Delta^0_2 \Rightarrow \mathcal{I}C = \Delta^0_1$, if taken together with certain additional assumptions.

**Mostowski’s assumptions**

1. There is a recursive enumeration of finite models,
2. Every finite model $\mathbb{N}_m$ has a recursive satisfaction relation.

The main assumptions of Mostowski’s argument taken together with the FM-representability theorem are actually equivalent to a version of the Learnability Thesis. Why?
Why testing formulae?

Testing formulae enabled us to distinguish relations that can be effectively verified (not only described) in potentially infinite domain with **computable** satisfaction relation.
Why testing formulae?

Testing formulae enabled us to distinguish relations that can be effectively verified (not only described) in potentially infinite domain with **computable** satisfaction relation.

It shall be a class of finite models such that *meaningful* concepts are computed in the limit.
Why testing formulae?

Testing formulae enabled us to distinguish relations that can be effectively verified (not only described) in potentially infinite domain with **computable** satisfaction relation.

It shall be a class of finite models such that *meaningful* concepts are computed in the limit.

In particular, such semantics gives us a class of formulae *decidable in the limit*. Such formulae express exactly intuitively learnable concepts. By the FM-representability theorem the set of such concepts is identical to the set of $\Delta^0_2$ relations.
Question: can we justify the Church’s Thesis by Learnability Thesis?
Question: can we justify the Church’s Thesis by Learnability Thesis? We expand our vocabulary to $\sigma' = \sigma \cup \{A\}$, where $A$ is an additional one-place predicate.
Question: can we justify the Church’s Thesis by Learnability Thesis?

We expand our vocabulary to $\sigma' = \sigma \cup \{A\}$, where $A$ is an additional one-place predicate.

**Theorem**

Let $(\mathbb{N}, A)$ be any $\sigma'$-model, $R \subseteq \omega^n$. $R$ is decidable in $A$ if and only if there are $\sigma'$-formulae $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$ such that $\psi(x_1, \ldots, x_n)$ is a testing formula in $\text{FM}((\mathbb{N}, A))$ for $\varphi(x_1, \ldots, x_n)$ and $R$. 
Question: can we justify the Church’s Thesis by Learnability Thesis? We expand our vocabulary to $\sigma' = \sigma \cup \{A\}$, where $A$ is an additional one-place predicate.

**Theorem**

Let $(\mathbb{N}, A)$ be any $\sigma'$-model, $R \subseteq \omega^n$. $R$ is decidable in $A$ if and only if there are $\sigma'$-formulae $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$ such that $\psi(x_1, \ldots, x_n)$ is a testing formula in $FM((\mathbb{N}, A))$ for $\varphi(x_1, \ldots, x_n)$ and $R$.

Taking $FM(\mathbb{N})$ as our formal model is aimed at distinguishing exactly those properties that are essential for performing intuitive computations.
Relativisation

The arithmetical hierarchy can be naturally relativised to capture notions concerning computations relative to oracles.

A relation $P$ is e.g. $\Delta^A_2$ if it is definable both by $\Sigma^A_2$ and $\Pi^A_2$ formulae i.e.:

$P(a) \equiv \exists x \forall y R(x, y, a)$

$P(a) \equiv \forall x \exists y S(x, y, a)$

for some recursive in $A$ predicates $R$ and $S$. 
Relativisation

The arithmetical hierarchy can be naturally relativised to capture notions concerning computations relative to oracles.

A relation $P$ is e.g. $\Delta^A_2$ if it is definable both by $\Sigma^A_2$ and $\Pi^A_2$ formulae i.e.:

$$P(a) \equiv \exists x \forall y R(x, y, a),$$

$$P(a) \equiv \forall x \exists y S(x, y, a),$$

for some recursive in $A$ predicates $R$ and $S$. 

Czarnecki, Godziszewski, Kalociński
Learnability and Church’s Thesis
CUNY LW 2016
Relativisation

The arithmetical hierarchy can be naturally relativised to capture notions concerning computations relative to oracles. A relation $P$ is e.g. $\Delta^A_2$ if it is definable both by $\Sigma^A_2$ and $\Pi^A_2$ formulae i.e.:

\[
P(\bar{a}) \equiv \exists x \forall y \ R(x, y, \bar{a}),
\]

\[
P(\bar{a}) \equiv \forall x \exists y \ S(x, y, \bar{a}),
\]

for some recursive in $A$ predicates $R$ and $S$. 
Theorem

Let $R \subseteq \omega^n$. The following are equivalent:

- $R$ is $\text{FM}$-representable in $\text{FM}(N, A)$,
- $R$ is $\Delta^A_2$.

Proof.
The theorem is obvious by the relativisation of the Limit Lemma.
Let $R \subseteq \omega^n$. The following are equivalent:

- $R$ is FM-representable in $\mathsf{FM}(\mathbb{N}, A)$,
Theorem

Let $R \subseteq \omega^n$. The following are equivalent:

- $R$ is FM-representable in $FM(\mathbb{N}, A)$,
- $R$ is $\Delta^A_2$.

Proof.
The theorem is obvious by the relativisation of the Limit Lemma.
Relativisation

Theorem
Let $R \subseteq \omega^n$. The following are equivalent:

- $R$ is FM-representable in $\text{FM}(\mathbb{N}, A)$,
- $R$ is $\Delta^A_2$.

Proof.
The theorem is obvious by the relativisation of the Limit Lemma.
Lowness

Definition (Low sets)

Let $A \subseteq \omega$. $A$ is low if $\deg(A)' = 0'$.

Theorem

Let $A$ be a low set. Then $\Delta^A_2 = \Delta^0_2$.

Proof.

Fix a low set $A$. Then for some recursive predicates $R$ and $S$ we have:

$P(a) \equiv \exists x \forall y R(x, y, a) \leq \deg(A)'$ and $P(a) \equiv \forall x \exists y S(x, y, a) \leq \deg(A)'$.

Since $A$ is low, $\deg(A)' = 0'$. Therefore $P$ is recursive in $0'$ and thus, by the Limit Lemma, $P$ is $\Delta^0_2$. 
Lowness

**Definition (Low sets)**

Let $A \subseteq \omega$ be a low set if $\deg(A)' = 0'$.
**Definition (Low sets)**

Let \( A \subseteq \omega \) be low if \( \text{deg}(A)' = 0' \).

**Theorem**

Let \( A \) be a low set. Then \( \Delta_2^A = \Delta_2^0 \).
Definition (Low sets)

Let $A \subseteq \omega$ be low if $\text{deg}(A)' = 0'$.

Theorem

Let $A$ be a low set. Then $\Delta^A_2 = \Delta^0_2$.

Proof.

Fix a low set $A$. and fix a $\Delta^A_2$ relation $P$. Then for some recursive in $A$ predicates $R$ and $S$ we have: $P(\bar{a}) \equiv \exists x \forall y \; R(x, y, \bar{a})$ and $P(\bar{a}) \equiv \forall x \exists y \; S(x, y, \bar{a})$. Since $A$ is low, $\text{deg}(A)' = 0'$. Therefore $P$ is recursive in $0'$ and thus, by the Limit Lemma, $P$ is $\Delta^0_2$. $\Box$
Corollary

Let $A$ be a low set and $R \subseteq \omega^n$. The following are equivalent:

- $R$ is FM-representable in $\text{FM}((\mathbb{N}, A))$,
- $R$ is $\Delta^0_2$.
Let $A$ be a low set and $R \subseteq \omega^n$. The following are equivalent:

- $R$ is FM-representable in $\text{FM}((\mathbb{N},A))$,
- $R$ is $\Delta^0_2$.

By the Corollary, adding any low set $A$ to the FM-domain does not affect the class of FM-representable relations and therefore the Learnability Thesis itself.
The negative answer

We are ready to prove our main theorem:
We are ready to prove our main theorem:

Theorem (M. Czarnecki, D. Kalociński, G.)

*The Learnability Thesis does not entail the Church’s Thesis.*
We are ready to prove our main theorem:

**Theorem (M. Czarnecki, D. Kalociński, G.)**

*The Learnability Thesis does not entail the Church’s Thesis.*

**Proof.**

Let \( A \) be a low, non-recursive set. Let the interpretation of \( IC \) be \( \{ R : R \leq_T A \} \).

Therefore in such a model the Church’s Thesis fails. On the other hand, consider an FM-domain \( FM((\mathbb{N}, A)) \). We may consider such an FM-domain since \( A \in IC \).

By the previous Corollary, relations FM-representable in \( FM((\mathbb{N}, A)) \) are exactly those which are \( \Delta_0^2 \).

Therefore the Learnability Thesis holds in such a model.

We have shown that there is an interpretation of \( IC \) such that \( IC \neq \Delta_0^1 \) and \( IL = \Delta_0^2 \).

Therefore the Learnability Thesis does not entail Church’s Thesis.
The negative answer

We are ready to prove our main theorem:

**Theorem (M. Czarnecki, D. Kalociński, G.)**

The Learnability Thesis does not entail the Church’s Thesis.

**Proof.**

Let $A$ be a low, non-recursive set. Let the interpretation of $\mathcal{IC}$ be $\{R : R \leq_T A\}$. Therefore in such model the Church’s Thesis fails.
The negative answer

We are ready to prove our main theorem:

Theorem (M. Czarnecki, D. Kalociński, G.)

The Learnability Thesis does not entail the Church’s Thesis.

Proof.

Let $A$ be a low, non-recursive set. Let the interpretation of $\mathcal{IC}$ be $\{R : R \leq_T A\}$. Therefore in such model the Church’s Thesis fails.

On the other hand consider an FM-domain $\text{FM}((\mathbb{N}, A))$. We may consider such an FM-domain since $A \in \mathcal{IC}$.
We are ready to prove our main theorem:

**Theorem (M. Czarnecki, D. Kalociński, G.)**

*The Learnability Thesis does not entail the Church’s Thesis.*

**Proof.**

Let $A$ be a low, non-recursive set. Let the interpretation of $\mathcal{IC}$ be $\{R : R \leq_T A\}$. Therefore in such model the Church’s Thesis fails.

On the other hand consider an FM-domain $\text{FM}((\mathbb{N}, A))$. We may consider such an FM-domain since $A \in \mathcal{IC}$. By the previous Corollary, relations FM-representable in $\text{FM}((\mathbb{N}, A))$ are exactly those which are $\Delta^0_2$. Therefore the Learnability Thesis holds in such a model. We have shown that there is an interpretation of $\mathcal{IC}$ such that $\mathcal{IC} \neq \Delta^0_1$ and $\text{IL} = \Delta^0_2$. Therefore the Learnability Thesis does not entail Church’s Thesis.
The negative answer

We are ready to prove our main theorem:

Theorem (M. Czarnecki, D. Kalociński, G.)

*The Learnability Thesis does not entail the Church’s Thesis.*

Proof.

Let $A$ be a low, non-recursive set. Let the interpretation of $\mathcal{I}C$ be \{$R : R \leq_T A$\}. Therefore in such model the Church’s Thesis fails.

On the other hand consider an FM-domain $FM((\mathbb{N}, A))$. We may consider such an FM-domain since $A \in \mathcal{I}C$. By the previous Corollary, relations FM-representable in $FM((\mathbb{N}, A))$ are exactly those which are $\Delta^0_2$. Therefore the Learnability Thesis holds in such a model.
The negative answer

We are ready to prove our main theorem:

**Theorem (M. Czarnecki, D. Kalociński, G.)**

*The Learnability Thesis does not entail the Church’s Thesis.*

**Proof.**

Let $A$ be a low, non-recursive set. Let the interpretation of $\mathcal{IC}$ be $\{ R : R \leq_T A \}$. Therefore in such model the Church’s Thesis fails.

On the other hand consider an FM-domain $\text{FM}((\mathbb{N}, A))$. We may consider such an FM-domain since $A \in \mathcal{IC}$. By the previous Corollary, relations FM-representable in $\text{FM}((\mathbb{N}, A))$ are exactly those which are $\Delta^0_2$. Therefore the Learnability Thesis holds in such a model.

We have shown that there is an interpretation of $\mathcal{IC}$ such that $\mathcal{IC} \neq \Delta^0_1$ and $\mathcal{IL} = \Delta^0_2$. 
The negative answer

We are ready to prove our main theorem:

**Theorem (M. Czarnecki, D. Kalociński, G.)**

*The Learnability Thesis does not entail the Church’s Thesis.*

**Proof.**

Let $A$ be a low, non-recursive set. Let the interpretation of $\mathcal{IC}$ be $\{ R : R \leq^T A \}$. Therefore in such model the Church’s Thesis fails.

On the other hand consider an FM-domain $\mathcal{FM}((\mathbb{N}, A))$. We may consider such an FM-domain since $A \in \mathcal{IC}$. By the previous Corollary, relations FM-representable in $\mathcal{FM}((\mathbb{N}, A))$ are exactly those which are $\Delta^0_2$. Therefore the Learnability Thesis holds in such a model.

We have shown that there is an interpretation of $\mathcal{IC}$ such that $\mathcal{IC} \neq \Delta^0_1$ and $\mathcal{IL} = \Delta^0_2$. Therefore the Learnability Thesis does not entail Church’s Thesis. 

□
Other interpretations of $\mathcal{IL}$

**Definition (Turing ideal)**

A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a Turing ideal if for all $X, Y \subseteq \omega$:

- $Y \in \mathcal{A} \land X \leq_T Y \Rightarrow X \in \mathcal{A}$,
- $X \oplus Y \in \mathcal{A}$.
Other interpretations of $\mathcal{L}$

**Definition (Turing ideal)**

A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a Turing ideal if for all $X, Y \subseteq \omega$:

- $Y \in \mathcal{A} \land X \leq_T Y \Rightarrow X \in \mathcal{A}$,
- $X \oplus Y \in \mathcal{A}$.

**Theorem**

There is a Turing ideal consisting solely of low sets.
A family $A \subseteq \mathcal{P}(\omega)$ is a Turing ideal if for all $X, Y \subseteq \omega$:

- $Y \in A \land X \leq_T Y \Rightarrow X \in A$,
- $X \oplus Y \in A$.

**Theorem**

There is a Turing ideal consisting solely of low sets.

**Theorem**

Let $\{A_i\}_{i \in \omega}$ be such that $\bigcup_{i \in \omega} \{A_i\}$ is a countable Turing ideal of low sets. Then $R$ is $FM(\mathbb{N}, \{A_i\}_{i \in \omega})$-representable iff $R$ is $\Delta^0_2$. 
Conclusion

Setting $\mathcal{IL} := 0 \cup A$, where $A$ is any Turing ideal of low sets, preserves the Learnability Thesis.
Diversity of interpretations of $\mathcal{IC}$

It is known that there are Turing ideals which are not generated by a single set.
Diversity of interpretations of $\mathcal{IC}$

It is known that there are Turing ideals which are not generated by a single set.
Take $A_0 <_T A_1 <_T A_2 <_T \ldots$ and $\mathcal{I} = \{B : \exists i B \leq_T A_i\}$.
Diversity of interpretations of $\mathcal{IC}$

It is known that there are Turing ideals which are not generated by a single set.

Take $A_0 <_T A_1 <_T A_2 <_T \ldots$ and $\mathcal{I} = \{B: \exists i B \leq_T A_i\}$.

It is easy to see that $\mathcal{I}$ is a Turing ideal and that no single set (or a finite family of sets) generates it.
Diversity of interpretations of IC

It is known that there are Turing ideals which are not generated by a single set.
Take $A_0 <_T A_1 <_T A_2 <_T \ldots$ and $\mathcal{I} = \{ B : \exists i B \leq_T A_i \}$. It is easy to see that $\mathcal{I}$ is a Turing ideal and that no single set (or a finite family of sets) generates it.

It is also known by Sacks density theorem that between every pair of recursively enumerable sets $A, B$ such that $A <_T B$ there are incomparable recursively enumerable sets $C, D$ i.e. $A <_T C <_T B$, $A <_T D <_T B$ and $C \perp D$. 
Diversity of interpretations of $\mathcal{IC}$

It is known that there are Turing ideals which are not generated by a single set.

Take $A_0 <_T A_1 <_T A_2 <_T \ldots$ and $\mathcal{I} = \{ B : \exists i \ B \leq_T A_i \}$.

It is easy to see that $\mathcal{I}$ is a Turing ideal and that no single set (or a finite family of sets) generates it.

It is also known by Sacks density theorem that between every pair of recursively enumerable sets $A, B$ such that $A <_T B$ there are incomparable recursively enumerable sets $C, D$ i.e. $A <_T C <_T B, A <_T D <_T B$ and $C \perp D$.

In fact we can have that $C \oplus D \equiv_T B$. 
Diversity of interpretations of $\mathcal{IC}$

Therefore let $A = A_\varepsilon$ and $B$ be recursively enumerable sets. Iterating the result by Sacks we can obtain a full binary tree of degrees below $B$ such that for every $\sigma \in \{0, 1\}^*$ it holds that $A_{\sigma_0} \oplus A_{\sigma_1} \equiv_T B$ and $A_{\sigma_0} \perp A_{\sigma_1}$. It follows that for and $\sigma, \tau \in \{0, 1\}^*$ the fact that $A_{\sigma} \perp A_{\tau}$ is equivalent to the fact that $\sigma$ and $\tau$ are incomparable. Thus each infinite branch encodes an increasing sequence of recursively enumerable sets that generates a Turing ideal. If on the start we take $B$ low, then every $A_{\sigma}$ is low and the ideals generated by infinite branches consist of low degrees only. It follows that there is a continuum of Turing ideals of low sets, therefore the choice for the interpretation of $\mathcal{IC}$ is indeed wide.
Diversity of interpretations of $\mathcal{IC}$

Therefore let $A = A_\epsilon$ and $B$ be recursively enumerable sets.
Iterating the result by Sacks we can obtain a full binary tree of degrees below $B$ such that for every $\sigma \in \{0, 1\}^*$ it holds that $A_{\sigma 0} \oplus A_{\sigma 1} \equiv_T B$ and $A_{\sigma 0} \perp A_{\sigma 1}$.
It follows that for and $\sigma, \tau \in \{0, 1\}^*$ the fact that $A_\sigma \perp A_\tau$ is equivalent to the fact that $\sigma$ and $\tau$ are incomparable.
Diversity of interpretations of $\mathcal{IC}$

Therefore let $A = A_\varepsilon$ and $B$ be recursively enumerable sets. Iterating the result by Sacks we can obtain a full binary tree of degrees below $B$ such that for every $\sigma \in \{0, 1\}^*$ it holds that $A_{\sigma_0} \oplus A_{\sigma_1} \equiv_T B$ and $A_{\sigma_0} \perp A_{\sigma_1}$.

It follows that for and $\sigma, \tau \in \{0, 1\}^*$ the fact that $A_{\sigma} \perp A_{\tau}$ is equivalent to the fact that $\sigma$ and $\tau$ are incomparable.

Thus each infinite branch encodes an increasing sequence of recursively enumerable sets that generates a Turing ideal. If on the start we take $B$ low, then every $A_{\sigma}$ is low and the ideals generated by infinite branches consist of low degrees only.
Diversity of interpretations of $\mathcal{IC}$

Therefore let $A = A_{\varepsilon}$ and $B$ be recursively enumerable sets. Iterating the result by Sacks we can obtain a full binary tree of degrees below $B$ such that for every $\sigma \in \{0, 1\}^*$ it holds that $A_{\sigma_0} \oplus A_{\sigma_1} \equiv_T B$ and $A_{\sigma_0} \perp A_{\sigma_1}$.

It follows that for and $\sigma, \tau \in \{0, 1\}^*$ the fact that $A_{\sigma} \perp A_{\tau}$ is equivalent to the fact that $\sigma$ and $\tau$ are incomparable.

Thus each infinite branch encodes an increasing sequence of recursively enumerable sets that generates a Turing ideal. If on the start we take $B$ low, then every $A_{\sigma}$ is low and the ideals generated by infinite branches consist of low degrees only.

It follows that there is a continuum of Turing ideals of low sets, therefore the choice for the interpretation of $\mathcal{IC}$ is indeed wide.
An attempt of justifying the Church's Thesis based only on the Learnability Thesis must fail. The clue of the argument is that there exists an interpretation of intuitive computability consistent with the Learnability Thesis such that certain intuitively computable sets are by no means recursive. If we admit certain non-recursive but intuitively computable relations (namely: some low relations) we are able to consider expanded FM-domains. On the other hand, by the Corollary, relations FM-representable in such FM-domains are still $\Delta^0_2$. The choice of interpretations of $IC$ is actualy very wide.
Conclusions II

An attempt of justifying the Church’s Thesis based only on the Learnability Thesis must fail.
An attempt of justifying the Church’s Thesis based only on the Learnability Thesis must fail.

The clue of the argument is that there exists an interpretation of intuitive computability consistent with the Learnability Thesis such that certain intuitively computable sets are by no means recursive.
An attempt of justifying the Church’s Thesis based only on the Learnability Thesis must fail.
The clue of the argument is that there exists an interpretation of intuitive computability consistent with the Learnability Thesis such that certain intuitively computable sets are by no means recursive.
If we admit certain non-recursive but intuitively computable relations (namely: some low relations) we are able to consider expanded FM-domains. On the other hand, by the Corollary, relations FM-representable in such FM-domains are still $\Delta^0_2$. 
An attempt of justifying the Church’s Thesis based only on the Learnability Thesis must fail.
The clue of the argument is that there exists an interpretation of intuitive computability consistent with the Learnability Thesis such that certain intuitively computable sets are by no means recursive.
If we admit certain non-recursive but intuitively computable relations (namely: some low relations) we are able to consider expanded FM-domains. On the other hand, by the Corollary, relations FM-representable in such FM-domains are still $\Delta^0_2$.
The choice of interpretations of \( \mathcal{IC} \) is actually very wide.
Thank you