Determinacy within Second Order Arithmetic

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Determinacy in Second Order Arithmetic, Z_2

- Part I: Determinacy in $\mathsf{Z_2}$, Second Order Arithmetic.
- Part 2: Determinacy of some similar larger classes within Δ^1_2.

Locus Classicus

Theorem (H Friedman -71, Martin)

$Z_2 \nvdash Det(\Sigma^0_4)$

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Definition (α-Γ)

Let Γ be a pointclass (*e.g.* $\Pi^0_3, \Pi^1_1, \ldots$). Let $\alpha < \omega_1$.
Then a set $B \subseteq \mathbb{R}$ is α-Γ if there is a sequence of Γ-sets $\langle A_\beta \mid \beta \leq \alpha \rangle$ with $A_\alpha = \emptyset$ so that:

$$x \in B \leftrightarrow \mu\beta(x \notin A_\beta) \text{ is odd .}$$

Theorem (Kuratowski 58)

$$\mathbf{\Delta}^{\mathbf{0}}_{\boldsymbol{n}+\mathbf{1}} = \bigcup_{\alpha<\omega_1} \alpha\text{-}\mathbf{\Pi}^{\mathbf{0}}_{\boldsymbol{n}}$$

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$$\mathsf{Z}_2 \vdash \mathit{Det}(n\text{-}\Pi^0_3)$$

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• Let β_0 be least so that $\omega^\omega \cap L_{\beta_0} \models \mathsf{Z_2}$.
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Then also β_0 is least so that $L_{\beta_0} \models ZF^-$.
• Let α_n be least so that $\omega^\omega \cap L_{\alpha_n} \models \Pi^1_{n+1}\text{-}\mathsf{CA_0}$.
Then also α_n is least so that $L_\alpha \models \Sigma_n\text{-}KP$.

Reverse Mathematical table

Table: Strengths of Determinacy

Γ	$Det(\Gamma)$	over Base Theory	
Δ^0_1	ATR_0	RCA_0	Steel 78
Σ^0_1	ATR_0	RCA_0	Steel 78
$\Sigma^0_1 \wedge \Pi^0_1$	Π^1_1-CA_0	RCA_0	Tanaka 90
Δ^0_2	Π^1_1-TR_0	RCA_0	Tanaka 91
Σ^0_2	Σ^1_1-ID_0	ATR_0	Tanaka 91
Δ^0_3	$[\Sigma^1_1]^{TR}$-ID_0	Π^1_1-TR_0	Med-Salem,Tanaka 08

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n-Π^0_3	Π^1_{n+2}-$\mathsf{CA}_0 \vdash \dots$ Δ^1_{n+2}-$\mathsf{CA}_0 + \dots \nvdash \dots$		Montalban-Shore 11

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$n\text{-}\Pi^0_3$	$\Pi^1_{n+2}\text{-}\mathsf{CA}_0 \vdash \ldots$ $\Delta^1_{n+2}\text{-}\mathsf{CA}_0 + \cdots \nvdash \ldots$		Montalban-Shore 11
Σ^0_4	$\mathsf{Z}_2 \nvdash \ldots$		Friedman-71, Martin

Theorem (W-09)

The theories:

$$\Pi^1_3\text{-}\mathsf{CA}_0,\ \Delta^1_3\text{-}\mathsf{CA}_0 + \mathit{Det}(\Sigma^0_3),\ \Delta^1_3\text{-}\mathsf{CA}_0$$

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Whereas:

• $\omega^\omega \cap L_\alpha$ is the least β-model of $\Delta^1_{n+1}\text{-}\mathsf{CA}_0$ where L_α is the least Σ_n-admissible model,

one has

• $\omega^\omega \cap L_\alpha$ is the least β-model of $\Pi^1_{n+1}\text{-}\mathsf{CA}_0$ where L_α is the least Σ_n-*non-projectible* model.

Definition (infinitely Σ_2-nested)

(i) We say that γ is *infinitely* Σ_2*-nested* if L_γ is the wellfounded part of some non-wellfounded model $\mathcal{M}$ and there are $\zeta_n, c_n \in On^{\mathcal{M}} (n < \omega)$

$$\zeta_0 \leq \cdots \leq \zeta_n \leq \cdots < \gamma < \cdots c_n < \cdots < c_0 \text{ with}$$

$$L_{\zeta_n} \prec_{\Sigma_2} (L_{c_n})^{\mathcal{M}}.$$

Where Σ^0_3-strategies lie

- Let γ_0 be the least Σ_2 - nested ordinal.

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Theorem (W09)

$$\eta = \gamma_0.$$

- γ_0 is non-Σ_1-projectible, and is less than the first Σ_2-non-projectible.
- Let β be least with $L_\beta \prec_{\Sigma_1} L_{\gamma_0}$.

Theorem (W-09)

(i) $\forall\delta < \beta\exists$a Σ^0_3-game, which is a win for Player I with no winning strategy in L_δ.
(ii) Any Σ^0_3-game, which is a win for Player I has a winning strategy in L_β.

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- Fact: The Σ_1-$Th(L_\beta)$ is a complete $\Game\Sigma^0_3$-set.

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• Fact: The Σ_1-$Th(L_\beta)$ is a complete $\Game\Sigma^0_3$-set.

Theorem (W-09)

(i) Any Σ^0_3*-game, which is a win for Player II has a winning strategy definable over* L_{γ_0}*. Hence:*
(ii) γ_0 *is the least ordinal so that any* Σ^0_3 *has a winning strategy definable over* L_{γ_0}.

Conjecture: γ_0 is the closure ordinal of $\partial\Pi^0_3$-monotone inductions.

The Consistency of Z_2

Theorem (Montalban-Shore-11)

(i) $\mathsf{Z}_2 \nvdash \forall n < \omega \; Det(n\text{-}\Pi^0_3)$*).*
(ii) $\mathsf{RCA}_0 + Det(n\text{-}\Pi^0_3) \vdash$ *"there is a β-model of Δ^1_{n+1}-CA_0".*

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Theorem (Montalban-Shore-11)

$\mathsf{RCA}_0 + \forall n Det(n\text{-}\Pi^0_3) \vdash Con(\mathsf{Z}_2)$.

Larger classes

Q. What if instead of looking at the determinacy available in the least model of $V = L + ZF^-$, we looked at the least iterable model of $V = L^\mu + ZF^-$?

The context here are the results:

Theorem (Martin-70's)

Let $\alpha \neq \omega^2.\beta$. Thus suppose $\omega^2.\beta < \alpha < \omega^2.(\beta + 1) < \omega_1$. Then:

$ZFC \vdash Det(\alpha\text{-}\Pi^1_1) \leftrightarrow$
$\leftrightarrow \exists$ *a # for an IM("There exists a β sequence of measurable cardinals)."*

Some work in progress

Conjecture 1: If there is an iterable model of Σ_2*-Separation* $+V = L^\mu$ *then* *Det*$(\omega^2\text{-}\Pi^1_1 \star \Sigma^0_3)$.

where $\omega^2\text{-}\Pi^1_1 \star \Sigma^0_3$ is actually $\omega^2 + 1\text{-}\Pi^1_1$ but where we require that A_{ω^2} is a Σ^0_3 set.

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Conjecture 2: If there is an iterable model of $ZFC^- +V = L^\mu$ *then for every* $n < \omega$ *Det*$(\omega^2\text{-}\Pi^1_1 \star n\text{-}\Sigma^0_n)$.