An infinite chess position with game value $\omega^4$

Norman Lewis Perlmutter

LaGuardia Community College

Set Theory Day, March 2016
This talk and presentation feature joint work with:

C. D. A. Evans (U.S. national master)
Joel David Hamkins

What is an infinite game?

Consider two-player infinite games of perfect information. The two players take turns making moves:

White: \[ a_0, a_2, a_4, a_6, \ldots \]

Black: \[ a_1, a_3, a_5, \ldots \]

Together they build a particular play \( \vec{a} = \langle a_0, a_1, a_2, \ldots \rangle \) of the game.

White wins this instance of the game if the play \( \vec{a} \in A \), a fixed set specifying the winning conditions. Otherwise, Black wins.
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Winning strategies

A *strategy* for a player is a function that tells the player what move to make next, given the finite sequence of preceding moves.

\[ \sigma : \langle a_0, a_1, \ldots, a_{k-1} \rangle \mapsto a_k \]

A *winning* strategy for a player is a strategy that, when followed, results in a winning play, no matter how the opponent plays.

Obviously, at most one of the players can have a winning strategy.
Open games

Open games generalize the concept of finite games.

A game is open for a player, if every winning instance of the game for that player is won already at a finite stage. That is, if $\vec{a}$ is a winning play, then there is some finite play $s = \vec{a} \upharpoonright n$ such that all extensions of $s$ are winning, no matter how play continues from $s$.

This is equivalent to saying that $A$ is an open set in the product topology on the space of all plays.

Similarly, a game is closed for a player, if it is open for the opposing player.

To win an open game, you must win at some finite stage. To win a closed game, it suffices never to lose at any finite stage.
Open Determinacy

The Axiom of Determinacy states that every (countably infinite) game is determined. This axiom contradicts AC. However, in ZFC, the following can be proven.

**Theorem (Gale, Stewart 1953)**

Every open game is determined. One of the players has a winning strategy.

We give a proof using the concept of ordinal game values.
Game value

In any open game, we define the ordinal *game value* of positions:

- A position that is already won has value 0.
- A position one move away from being won has value 1.

The values continue transfinitely:

- value is $\alpha + 1$, if white may play to a position with value $\alpha$.
- if black to play, value is supremum of the values of the positions to which black may play.

Some positions may be left without value.

White can play to strictly decrease value. Black can maintain value, but not increase it.
Game values provide strategies

White wants to reduce value. From a valued position, white can do so.

Black wants to maintain value. He cannot increase it.

Conclusion
A position is winning for white if and only if it has an ordinal value.

White wins via the value-reducing strategy.
Black wins via the value-maintaining strategy.
**Illustrating game value $\omega^2$**

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An infinite chess position with game value $\omega^4$
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Concrete meaning of the game values

Finite value $n$.

O
O
. . . . white can force a win in $n$ moves, but not faster.

Value $\omega$.

O
O
. . . . black to move, but every move leads to some (arbitrarily large) finite value $n$. Black will lose, but he can cause an arbitrarily long delay.

Value $\omega + 10$.

O
O
. . . . white can force value $\omega$ in 10 moves.

Value $\omega^2$.

O
O
. . . . black to play, but every move creates value $\omega \cdot n + k$. Think: $n$ is the number of future announcements he will make, for each of which play will take that long before the next announcement.

Value $\omega^3$.

O
O
. . . . black can announce an arbitrary number, which will be the number of large announcements. Each large announcement is the number of small announcements to be made; each small announcement is the number of moves before the next announcement.
Infinite chess

Infinite chess is chess played on an infinite edgeless chess board, arranged like the integer lattice $\mathbb{Z} \times \mathbb{Z}$.

The familiar chess pieces—kings, queens, bishops, knights, rooks and pawns—move about according to their usual chess rules, with bishops on diagonals, rooks on ranks and files and so on, with each player striving to place the opposing king into checkmate.

There is no standard starting configuration in infinite chess, but rather a game proceeds by setting up a particular position on the board and then playing from that position.
Clarifying the rules

Let me clarify a few of the rules as they relate to infinite chess.

- At most one king of each color
- There is no boundary, hence no pawn promotion
- There is no castling and no *en passant*
- Abandon the 50 move rule as limiting
- Infinite play is a draw
- We may abandon the three-fold repetition rule
- For arbitrary starting positions, one must clarify some weird boundary cases, e.g. White to move, black in check, white in checkmate
Infinite chess is an open game

Checkmate, when it occurs, does so after finitely many moves. Infinite chess is therefore an open game, and thus subject to the theory of transfinite game values.

So we can imagine chess positions with various large ordinal game values.

(The possibility of draw does not significantly upset the transfinite game value analysis.)
The omega one of chess

\( \omega_1^{ch} \) is defined as the supremum of all game values corresponding to positions of infinite chess. We know that

\[ \omega^4 \leq \omega_1^{ch} \leq \omega_1. \]

**Conjecture (Evans, Hamkins)**
The omega one of chess is as large as it could be.

\[ \omega_1^{ch} = \omega_1. \]

(proven in 3 dimensions)

**Conjecture (Perlmutter)**
The omega one of chess is not as large as it could be.

\[ \omega_1^{ch} < \omega_1. \]
A finite position with value $\omega$
A finite position with value $\omega$

Black moves up arbitrary height
A finite position with value $\omega$

Check

An infinite chess position with game value $\omega^4$
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An infinite chess position with game value $\omega^4$
A finite position with value $\omega$

Checkmate. Black can cause arbitrary delay, but the only choice is on first move, so the initial value is $\omega$. 
Several infinite positions with value $\omega$
Positions with value $\omega^2$
Releasing the Hordes, with value $\omega^2$

Black to move.
Releasing the Hordes, with value $\omega^2$

He moves trapped rook up arbitrary height.

An infinite chess position with game value $\omega^4$
Infinite games and transfinite game values

Transfinite game values in infinite chess

Releasing the Hordes, with value $\omega^2$

White should capture from left side.

An infinite chess position with game value $\omega^4$
Releasing the Hordes, with value $\omega^2$

Now black begins to harass white king.
Releasing the Hordes, with value $\omega^2$

White must chase down the rook to avoid perpetual check.
Releasing the Hordes, with value $\omega^2$
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Black must move away to save rook.
Now is white’s chance to advance a pawn.
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Black moves arbitrary distance out.
Releasing the Hordes, with value $\omega^2$

Another chance to advance a pawn.
Releasing the Hordes, with value $\omega^2$

Black harasses the white king.
Releasing the Hordes, with value $\omega^2$

White must chase him down.
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Infinite games and transfinite game values

Transfinite game values in infinite chess

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(Black should actually move arbitrary distance to the right.)
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Releasing the Hordes, with value $\omega^2$
Infinite games and transfinite game values

Transfinite game values in infinite chess

Releasing the Hordes, with value $\omega^2$

An infinite chess position with game value $\omega^4$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$

The bishop unlocks the door.
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$

An infinite chess position with game value $\omega^4$
Releasing the Hordes, with value $\omega^2$

An infinite chess position with game value $\omega^4$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$

An infinite chess position with game value $\omega^4$
Releasing the Hordes, with value $\omega^2$

Black can move rook arbitrary distance.
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$

An infinite chess position with game value $\omega^4$
Releasing the Hordes, with value $\omega^2$
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An infinite chess position with game value $\omega^4$
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Releasing the Hordes, with value $\omega^2$

The portcullis opens...
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
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Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$
Infinite games and transfinite game values

Releasing the Hordes, with value $\omega^2$

An infinite chess position with game value $\omega^4$
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Releasing the Hordes, with value $\omega^2$

Queens enter the mating chamber.
Releasing the Hordes, with value $\omega^2$
Releasing the Hordes, with value $\omega^2$

Checkmate

An infinite chess position with game value $\omega^4$
Iterated lock and key, value $\omega^2 \cdot 4$

Black to move.

An infinite chess position with game value $\omega^4$
The first black tower ascends.
Iterated lock and key, value $\omega^2 \cdot 4$
Iterated lock and key, value $\omega^2 \cdot 4$

Black harasses white king.

An infinite chess position with game value $\omega^4$
Iterated lock and key, value $\omega^2 \cdot 4$

An infinite chess position with game value $\omega^4$
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Iterated lock and key, value $\omega^2 \cdot 4$

Black should actually move arbitrary distance right.
Iterated lock and key, value $\omega^2 \cdot 4$

White advances a pawn.

An infinite chess position with game value $\omega^4$
Infinite games and transfinite game values

Transfinite game values in infinite chess

Iterated lock and key, value $\omega^2 \cdot 4$

An infinite chess position with game value $\omega^4$
Iterated lock and key, value $\omega^2 \cdot 4$
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An infinite chess position with game value $\omega^4$
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Black should actually move arbitrary distance right.
Iterated lock and key, value $\omega^2 \cdot 4$

White advances pawn.
Iterated lock and key, value $\omega^2 \cdot 4$
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An infinite chess position with game value $\omega^4$
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Iterated lock and key, value $\omega^2 \cdot 4$

White advances pawn.
Iterated lock and key, value $\omega^2 \cdot 4$
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An infinite chess position with game value $\omega^4$
Infinite games and transfinite game values

Transfinite game values in infinite chess

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Black should actually move arbitrary distance right.
Iterated lock and key, value $\omega^2 \cdot 4$

White advances pawn.

An infinite chess position with game value $\omega^4$
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White advances pawn.
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Transfinite game values in infinite chess

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Transfinite game values in infinite chess

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Infinite games and transfinite game values

Transfinite game values in infinite chess

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An infinite chess position with game value $\omega^4$
Iterated lock and key, value $\omega^2 \cdot 4$
Infinite games and transfinite game values

Transfinite game values in infinite chess

Iterated lock and key, value $\omega^2 \cdot 4$

Black should actually move arbitrary distance right.

An infinite chess position with game value $\omega^4$
Iterated lock and key, value $\omega^2 \cdot 4$

Black key pawn attacks second tower.
Iterated lock and key, value $\omega^2 \cdot 4$

Second tower ascends.
Iterated lock and key, value $\omega^2 \cdot 4$
Iterated lock and key, value $\omega^2 \cdot 4$
Infinite games and transfinite game values

Transfinite game values in infinite chess

Iterated lock and key, value $\omega^2 \cdot 4$

An infinite chess position with game value $\omega^4$
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Black should actually move arbitrary distance right.
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An infinite chess position with game value $\omega^4$
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An infinite chess position with game value $\omega^4$
Transfinite game values in infinite chess

Iterated lock and key, value $\omega^2 \cdot 4$

An infinite chess position with game value $\omega^4$
Iterated lock and key, value $\omega^2 \cdot 4$

An infinite chess position with game value $\omega^4$
Iterated lock and key, value $\omega^2 \cdot 4$

Black should actually move arbitrary distance right.
Previous state of the art: value $\omega^3$
A position with game value $\omega^4$

Next, I will show the position for game value $\omega^4$ in a separate file.
Thank you.

Some articles on infinite chess:

- N. Perlmutter, C.D.A. Evans, J.D. Hamkins, “A position in infinite chess with game value $\omega^4$,” manuscript under review.


Norman Lewis Perlmutter
LaGuardia Community College