Lattices of Elementary Substructures

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Lattices

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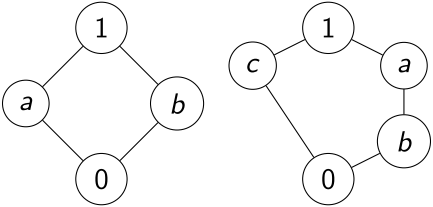

Lattices, II

Definition

Let L be a lattice.

- We say $a \in L$ is compact if whenever $X \subseteq L$ and $a \leq \bigvee X$, there is a finite $Y \subseteq X$ such that $a \leq \bigvee Y$.
- L is algebraic if it is complete (for all $X \subseteq L, \bigvee X$ and $\bigwedge X$ exist), and each element of L is the supremum of a set of compact elements.
- Let κ be an infinite cardinal. L is κ-algebraic if it is algebraic and, whenever $x \in L$ is compact, $|\{a \in L \mid a \leq x \text{ and } a \text{ is compact }\}| < \kappa$.

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Substructure and Interstructure Lattices

Given a model $\mathcal{M} \models \mathrm{PA}$, the set of all $\mathcal{K} \preceq \mathcal{M}$ forms a lattice under the inclusion relation. Additionally, if $\mathcal{M} \prec \mathcal{N}$, then the set of all $\mathcal{K}$ such that $\mathcal{M} \preceq \mathcal{K} \preceq \mathcal{N}$ also forms a lattice.

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Given a model $\mathcal{M} \models \mathrm{PA}$, the set $\{\mathcal{K} \models \mathrm{PA} \mid \mathcal{K} \preceq \mathcal{M}\}$ is called the substructure lattice of $\mathcal{M}$ and is denoted $\mathrm{Lt}(\mathcal{M})$. If $\mathcal{M} \prec \mathcal{N}$, then the set $\{\mathcal{K} \mid \mathcal{M} \preceq \mathcal{K} \preceq \mathcal{N}\}$ is called the interstructure lattice (between $\mathcal{M}$ and $\mathcal{N}$) and is denoted $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$.

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Let $\mathcal{M} \models \mathsf{PA}$. Then the compact elements of $\mathrm{Lt}(\mathcal{M})$ are the finitely generated submodels.

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If $\mathcal{M} \prec \mathcal{N}$ *and* $|M| = \kappa$*, then* $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ *is* κ^+*-algebraic.*

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- Blass (1974): Every countable nonstandard $\mathcal{M}$ has a minimal cofinal extension.

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A lattice L is called distributive if whenever $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

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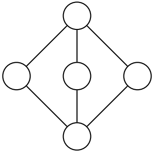

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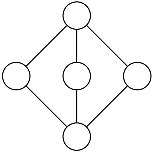

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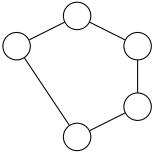

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The Lattice Problem

Question

For which lattices L is it the case that there is $\mathcal{M} \models \mathrm{PA}$ such that $\mathrm{Lt}(\mathcal{M}) \cong L$?

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Given a model $\mathcal{M} \models \mathrm{PA}$, for which lattices L is it the case that there is $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$?

Theorems are often of the form: "Every (countable)/(nonstandard) $\mathcal{M}$ has an elementary (end/cofinal) extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$".

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We also are interested in knowing the nature of these extensions: are they cofinal extensions, end extensions, can they be conservative, etc.

Example

There is no $\mathcal{M} \models$ TA such that $\text{Lt}(\mathcal{M}) \cong \mathbf{M}_3$.

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Representations of Lattices

Many proofs about interstructure lattices involve representations of lattices.

Definition

Let A be any set. Then Eq(A) is the set of equivalence relations on that set, and it forms a lattice under inclusion. $\mathbf{0_A}$ is the discrete relation, and $\mathbf{1_A}$ is the trivial relation.

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Let L be a finite lattice. $\alpha : L \to \mathrm{Eq}(A)$ is called a representation of L on A if:

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Let $\mathcal{M} \models \mathsf{PA}$ be countable, $A \subseteq M$ be unbounded and definable, and $\alpha : \mathbf{2} \to \mathrm{Eq}(A)$ be the (unique) representation.

If $\Theta \in \mathrm{Eq}(M)$ is definable, then there is an unbounded, definable $B \subseteq A$ such that either $\Theta \cap B^2 = \alpha(0) \cap B^2$ or $\Theta \cap B^2 = \alpha(1) \cap B^2$. (Every definable Skolem function can be made 1 to 1 or constant on an unbounded set.)

Enumerate the definable equivalence relations, apply the above statement to obtain a sequence $M = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$.

Let $p(x) = \{\phi(x) \mid \exists n < \omega\, \mathcal{M} \models \forall x \in A_n\, \phi(x)\}$. If c realizes $p(x)$, then $\mathcal{M} \prec_{\text{end}} \mathcal{M}(c)$ and $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{2}$.

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Let $\alpha : L \to \mathrm{Eq}(A)$ be a representation, and suppose $B \subseteq A$. Then $\alpha|B : L \to \mathrm{Eq}(B)$ is the function defined by $(\alpha|B)(r) = \alpha(r) \cap B^2$ for each $r \in L$.

Remark

$\alpha|B$ is not always a representation (for example, let $A = \mathbb{N}$, and $B = \{0\}$, then $(\alpha|B)(1) = (\alpha|B)(0)$).

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Let L be a finite lattice, and $\alpha : L \to \mathrm{Eq}(A)$ a representation. We say α is 0-CPP if for each $r > 0, \alpha(r)$ has more then 2 classes. We say α is $(n+1)$-CPP if for each $\Theta \in \mathrm{Eq}(A)$, there is $B \subseteq A$ such that $\alpha|B$ is an n-CPP representation and $r \in L$ such that $\Theta \cap B^2 = \alpha(r) \cap B^2$.

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Remark

If L has an n-CPP representation, then it has a finite n-CPP representation. This can be expressed in PA with a Σ_1 formula $cpp(L, n)$.

Theorem

Let $\mathcal{M}$ be a countable nonstandard model of PA^* and L a finite lattice. If $\mathcal{M} \models cpp(L, n)$ for each $n \in \omega$, then $\mathcal{M}$ has a cofinal extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

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Let $\mathcal{M}$ be a countable nonstandard model of PA^* *and L a finite lattice. If $\mathcal{M} \models cpp(L, n)$ for each $n \in \omega$, then $\mathcal{M}$ has a cofinal extension $\mathcal{N}$ such that* $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

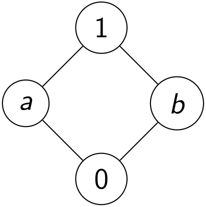
Theorem

Let L be a finite lattice, $\mathcal{M} \models \text{PA}^*$, and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$. Then $\mathcal{M} \models cpp(L, n)$ for each $n \in \omega$.

The Lattice B_2

Theorem

Let $\mathcal{M} \models \mathsf{PA}$ be countable (standard or non-standard). Then $\mathcal{M}$ has an elementary end extension $\mathcal{N}$ such that $\mathsf{Lt}(\mathcal{N}/\mathcal{M}) \cong \boldsymbol{B}_2$.



Representation of $\mathbf{B}_2$

Let $A = \{(x, y) \mid x \geq y\}$. Define a representation $\alpha : \mathbf{B}_2 \to \mathrm{Eq}(A)$ by letting $\langle (x_0, y_0), (x_1, y_1) \rangle \in \alpha(a)$ iff $x_0 = x_1$, and $\langle (x_0, y_0), (x_1, y_1) \rangle \in \alpha(b)$ iff $y_0 = y_1$.

We need a combinatorial lemma, similar to the one for the construction of a minimal extension.

Lemma

Let $\Theta \in \mathrm{Eq}(A)$ be definable. Then there is $B \subseteq A$ such that $\alpha|B \cong \alpha$ and $r \in \mathbf{B}_2$ such that $\Theta \cap B^2 = \alpha(r) \cap B^2$.

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Representation of B_2

Enumerate all the definable equivalence relations. Starting with $A_0 = A$, we obtain a sequence of definable sets $A_0 \supseteq A_1 \supseteq \ldots$ and we can construct a type
$p(x) = \{\phi(x) \mid \exists n < \omega\, \mathcal{M} \models \forall x \in A_n\, \phi(x)\}$

For $r \in \mathrm{B}_2$, let t_r be the Skolem term mapping x to the least y such that $\langle x, y \rangle \in \alpha(r)$.

Let c realize $p(x)$ and $\mathcal{N} = \mathcal{M}(c)$. Then $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathrm{B}_2$, with the other submodels $\mathcal{M}_a$ and $\mathcal{M}_b$ being generated by $t_a(c)$ and $t_b(c)$, respectively.

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Recent Application

Theorem (Schmerl 2014)

If $\mathcal{M}, \mathcal{N}$ are countable, arithmetically saturated models of PA, *and* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$, *then* $\mathrm{Th}(\mathcal{M})' \equiv_T \mathrm{Th}(\mathcal{N})'$.

The proof is very involved, so we mention part of one step here. To $X \subseteq \omega$, we associate an infinite distributive lattice $\mathcal{D}'(X)$.

Lemma

Let $\mathcal{M}$ be recursively saturated. Then there is $\mathcal{M}_1 \prec \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{M}_1) \cong \mathcal{D}'(X)$ *iff there is* $Y \in \mathrm{SSy}(\mathcal{M})$ *such that* $X \leq_T Y'$.

Using this (and a lot of power from the automorphism groups), we can then show that $\mathrm{Th}(\mathcal{M})' \leq_T \mathrm{Th}(\mathcal{N})'$.

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