Rigidity Properties of Precipitous Ideals

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Outline

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Generic Large Cardinals

Large cardinal properties

- witnessed by elementary embeddings $j : V \to M$ defined in V
- witnessed by ultrafilters $U \in V$

Generic large cardinal properties

- witnessed by external embeddings $j : V \to M$ defined in some extension $V[G]$
- witnessed by ideals (or filters) $I \in V$
- can hold at small cardinals ω_1, ω_2, etc.

Generic Ultrapowers

Review of Precipitous Ideals:
Suppose I is an ideal on a regular cardinal κ.

- For $X, Y \subseteq \kappa$ let $X \sim_I Y$ iff $X \triangle Y \in I$.
- We can define a forcing

$$\mathbb{P} = P(\kappa)/I - \{[\varnothing]\} = \{[X]_I : X \subseteq \kappa \wedge X \not\sim_I \varnothing\}$$

  where $[X]_I \leqslant [Y]_I$ iff $X \subseteq_I Y$.

Suppose G is V-generic for $P(\kappa)/I$.

- There is a V-ultrafilter $U_G \in V[G]$ extending the filter dual to I.
- The generic ultrapower $M = \mathrm{Ult}(V, U_G) = \{[f]_G : f \in V \wedge \mathrm{dom}(f) = \kappa\}$ may not be wellfounded.
- There is a generic ultrapower embedding $j_G : V \to M$ where $j_G(x) = [c_x]_G$.

Definition

A κ-complete ideal I on κ is **precipitous** if the corresponding generic ultrapower $j : V \to \mathrm{Ult}(V, U_G)$ is wellfounded.

Precipitousness

Theorem (Jech-Magidor-Mitchell-Prikry, 1980)

If κ is a measurable cardinal then there is a forcing extension $V[G]$ in which $\kappa = \omega_1^{V[G]}$ and there is a precipitous ideal on κ.

Proof.

- Let G be V-generic for $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$ and let j be an ultrapower by a normal measure on κ.
- $j(\mathbb{P}) = \mathrm{Col}(\omega, j(\kappa))^M \cong \mathrm{Col}(\omega, {<}\kappa) \times \mathrm{Col}(\omega, [\kappa, j(\kappa)))$
- Let H be $V[G]$-generic for $\mathrm{Col}(\omega, [\kappa, j(\kappa)))$.
- $j"G \subseteq H \times G$
- So j lifts to $j_H : V[G] \to M[G * H]$.
- j_H is the ultrapower by $U_H = \{X \in P(\kappa)^{V[G]} : \kappa \in j_H(X)\}$.
- $I = \{X \in P(\kappa)^{V[G]} : \Vdash_{j(\mathbb{P})/G} \kappa \notin j(X)\}$
- Why is I precipitous?
- In $V[G]$, forcing with $P(\kappa)^{V[G]}/I$ is equivalent to forcing with $j(\mathbb{P})/G$.

In $V[G]$, why is forcing with $P(\kappa)^{V[G]}/I$ equivalent to forcing with $j(\mathbb{P})/G$?

- Define a homomorphism of boolean algebras

$$\begin{aligned} i : & P(\kappa)^{V[G]} \to \mathrm{ro}(j(\mathbb{P})/G) \\ & X \mapsto \|\kappa \in j_{\dot{H}}(X)\|. \end{aligned}$$

- The kernel of i is I so we get

$$i : P(\kappa)^{V[G]}/I \to \mathrm{ro}(j(\mathbb{P})/G).$$

- The range of i is dense. □

The fact that forcing with $P(\kappa)^{V[G]}/I$ is equivalent to forcing with $j(\mathbb{P})/G$ is a special case of **Foreman's Duality Theorem**.

The Duality Theorem

Theorem (Foreman, 2013)

Suppose Z is a set and $\mathbb{P}$ is a forcing such that whenever $G \subseteq \mathbb{P}$ is V-generic, there is an ultrafilter U on Z such that V^Z/U is isomorphic to a transitive class M. Also assume that there are functions $f_{\mathbb{P}}, \langle f_p : p \in \mathbb{P}\rangle$ and g, such that

$$\Vdash_{\mathbb{P}} \text{“}[f_{\mathbb{P}}]_U = \mathbb{P} \wedge (\forall p \in \mathbb{P})[f_p]_U = p \wedge [g]_U = \dot{G}\text{”}.$$

If $I = \{X \in P(\mathbb{Z}) : \Vdash_{\mathbb{P}} [id]_{\dot{U}} \notin j_{\dot{U}}(X)\}$, then there is a dense embedding $d : P(Z)/I \to \mathrm{ro}(\mathbb{P})$.

- Forcing with the ideal derived from a generic ultrapower embedding (i.e. forcing with $P(\kappa)/I$) is in many cases equivalent to the forcing which adds the embedding.

Outline

Saturated and Presaturated Ideals

Definition

An ideal I on a regular cardinal κ is **saturated** if the boolean algebra $P(\kappa)/I$ has the κ^+-chain condition.

- saturation $\implies$ precipitousness.
- I saturated $\implies$ forcing with $P(\kappa)/I$ preserves κ^+.

Definition

An ideal I on a regular cardinal κ is **presaturated** if it is precipitous and forcing with $P(\kappa)/I$ preserves κ^+.

Rigid Ideals

$\mathbb{P}_{max}$ forcing - Woodin was concerned with homogeneity/rigidity properties of $P(\omega_1)/I_{NS}$.

Definition

An ideal I on a regular cardinal κ is **rigid** if forcing with $P(\kappa)/I$ produces an extension $V[G]$ in which there is a unique generic for $P(\kappa)/I$.

Theorem (Woodin, 1990s - Larson, 2002)

Assuming MA_{ω_1}, if I is a normal uniform saturated ideal on ω_1, then I is rigid.

- Of course $MA_{\omega_1} \implies \neg\mathrm{CH}$

Question

Is it consistent to have a normal uniform rigid saturated ideal on $\omega_1 + \mathrm{CH}$?

Outline

The Number of Normal Measures

$$0 \leqslant (\text{the number of normal measures on } \kappa) \leqslant 2^{2^\kappa}$$

Theorem (Kunen-Paris, 1971)

If κ is measurable then there is a forcing extension in which κ carries 2^{2^κ} normal measures.

Theorem (Apter-Hamkins-Cummings, 2007)

If κ is measurable then there is a forcing extension in which κ carries exactly κ^+ normal measures.

Theorem (Friedman-Magidor, 2009)

Assume GCH. Suppose κ is measurable and $\alpha \leqslant \kappa^{++}$ is a cardinal. Then in a cofinality-preserving forcing extension, κ carries exactly α normal measures.

Coding Forcing

Lemma (Friedman-Magidor, 2009)

Suppose κ is an inaccessible cardinal and G is V-generic for $\mathrm{Sacks}(\kappa)$. In $V[G]$ there is a forcing $\mathrm{Code}(\kappa)$ such that if H is $V[G]$-generic for $\mathrm{Code}(\kappa)$, then in $V[G * H]$ there is a unique V-generic filter for $\mathrm{Sacks}(\kappa) * \mathrm{Code}(\kappa)$.

The forcing $\mathrm{Code}(\kappa)$...

- adds a club C, and
- this club codes the Sack-generic G, as well as the Code-generic C into the stationarity/nonstationarity of sets in a sequence $\vec{E} = \langle E_\alpha : \alpha < \kappa^+ \rangle$ of ground model stationary sets.

Outline

Coding the Lévy-Collapse

Suppose κ is an inaccessible cardinal and let $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$.

Let G be a V-generic for $\mathbb{P}$. Then $\kappa = (\omega_1)^{V[G]}$.

Goal: Working in $V[G]$, define a forcing $\mathbb{Q}$ such that if H is $V[G]$-generic for $\mathbb{Q}$, then there is a unique V-generic filter for $\mathbb{P} * \mathbb{Q}$ in $V[G * H]$.

- This situation is similar to the Friedman-Magidor lemma.
- We want to code the Lévy-collapse generic into the stationarity/nonstationarity of sequence of sets.

Go back to working in V momentarily:

- Let $\vec{\eta} = \langle \eta_\alpha : \alpha < \kappa \rangle$ enumerate the uncountable regular cardinals less than κ.
- For each $\alpha < \kappa$, let $E_\alpha = \mathrm{cof}(\eta_\alpha) \cap (\eta_\alpha, \kappa)$.
- $\vec{E} = \langle E_\alpha : \alpha < \kappa \rangle$
- Fix a bijection $f : \kappa \to \mathrm{Col}(\omega, {<}\kappa)$ with $f \in V$.
- Let $W, X, Y, Z : \kappa \to \kappa$ be four cofinal functions with pairwise disjoint ranges.

Remark

In $V[G]$, each E_α is a stationary subset of $\kappa = \omega_1^{V[G]}$ since $\mathbb{P}$ is κ-c.c.

- $f : \kappa \to \mathrm{Col}(\omega, {<}\kappa)$ is a bijection
- $W, X, Y, Z : \kappa \to \kappa$ are cofinal functions with disjoint ranges.
- $\vec{E} = \langle E_\alpha : \alpha < \kappa \rangle$ seq. of stationary sets in $V[G]$.

Definition

Working in $V[G]$, conditions in $\mathbb{Q} = \mathrm{Code}(\kappa, \vec{E})$ are closed bounded subsets of $\kappa = \omega_1^{V[G]}$ ordered by $d \leqslant c$ iff

1. d is an end extension of c.
2. For $i < \kappa$, if $f(i) \in G$ then $d \setminus c$ is disjoint from $E_{W(i)}$ and if $f(i) \notin G$ then $d \setminus c$ is disjoint from $E_{X(i)}$.
3. For $i \leqslant \max(c)$, if $i \in c$ then $d \setminus c$ is disjoint from $E_{Y(i)}$ and if $i \notin c$ then $d \setminus c$ is disjoint from $E_{Z(i)}$.

- (2) codes G into the stationarity/nonstationarity of the sets in $\vec{E}$.
- (3) codes the generic for $\mathbb{Q} = \mathrm{Code}(\kappa, \vec{E})$ into the stationarity/nonstationarity of the sets in $\vec{E}$.
- $\mathrm{Code}(\kappa, \vec{E})$ is ${<}\kappa$-distributive in $V[G]$.

Lemma

Suppose κ *is inaccessible,* $G * H$ *is* V*-generic for* $\mathbb{P} * \mathbb{Q} = \mathrm{Col}(\omega, {<}\kappa) * \mathrm{Code}(\kappa, \vec{E})$ *and let* $C = \bigcup H$*. Then in* $V[G * H]$ *we have*

1. *For* $i < \kappa$, $f(i) \in G$ *iff* $E_{W(i)}$ *is nonstationary and* $f(i) \notin G$ *iff* $E_{X(i)}$ *is nonstationary.*
2. *For* $i < \kappa$, $i \in C$ *iff* $E_{Y(i)}$ *is nonstationary and* $i \notin C$ *iff* $E_{Z(i)}$ *is nonstationary.*
3. *There is a unique* V*-generic filter for* $\mathrm{Col}(\omega, {<}\kappa) * \mathrm{Code}(\kappa, \vec{E})$ *(in* $V[G * H]$*).*

- **Idea**: If there were multiple generics for $\mathrm{Col}(\omega, {<}\kappa) * \mathrm{Code}(\kappa, \vec{E})$ in $V[G * H]$, this would lead to an intermediate extension $V \subseteq V[G' * H'] \subseteq V[G * H]$ such that some set X is nonstationary in $V[G' * H']$ and stationary in $V[G * H]$.

A Rigid Precipitous Ideal + CH

Theorem (C.-Cox-Eskew, 2016)

Suppose κ is huge with target λ. There is a forcing extension in which there is a rigid precipitous ideal on $[\lambda]^{\omega_1}$ and GCH holds.

Proof.

- WLOG assume GCH and let $j : V \to M$ be an ultrapower witness that κ is huge with target λ.
- Let $G * H$ be V-generic for $\mathbb{P} * \mathbb{Q} = \mathrm{Col}(\omega, {<}\kappa) * \mathrm{Code}(\kappa, \vec{E})$.
- We will argue that there is a rigid precipitous ideal on $[\lambda]^{\kappa} = ([\lambda]^{\omega_1})^{V[G*H]}$.

$$I = \{X \in P([\lambda]^{\omega_1})^{V[G*H]} : \Vdash_{j(\mathbb{P})/i[G*H]*j(\mathbb{Q})/d} ([\mathrm{id}]_{\dot{U}} \notin j(X))\}$$

- We need to lift j to $V[G * H]$.

Extend j to $V[G]$:

- Absorption: there is a complete embedding $i : \mathbb{P} * \mathbb{Q} \to j(\mathbb{P}) = j(\mathrm{Col}(\omega, {<}j(\kappa)))$ extending the identity $\mathrm{id} : \mathbb{P} \to j(\mathbb{P})$.
- Let $\hat{G} \subseteq j(\mathbb{P})/i[G * H]$ be $V[G * H]$-generic.
- Then $V[G * H * \hat{G}]$ is an extension by $j(\mathbb{P})$.
- $j``G \subseteq \hat{G}$
- Thus j lifts to $j : V[G] \to M[\hat{G}]$.

Next lift j through $H \subseteq \mathbb{Q} = \mathrm{Code}(\kappa, \vec{E})$:

- $j[H] = H$ since conditions in $\mathbb{Q}$ have size less than the critical point of j.
- Clearly $d = (\bigcup j[H]) \cup \{\kappa\}$ is a condition in $j(\mathbb{Q})$.
- We must show that d is a master condition—i.e. that d is stronger than every element of $j[H]$.
- This will involve checking that adding κ to the top of $\bigcup j[H]$ doesn't cause us to hit a stationary set we are supposed to be avoiding.
- We have $j(\vec{\eta}) = \langle \bar{\eta}_\alpha : \alpha < j(\kappa) \rangle$ and $j(\vec{E}) = \langle \bar{E}_\alpha : \alpha < j(\kappa) \rangle$ where
- $\bar{E}_\alpha = [\mathrm{cof}(\bar{\eta}_\alpha) \cap (\bar{\eta}_\alpha, j(\kappa))]^V$ since $M^{j(\kappa)} \cap V \subseteq M$.
- We have $\kappa \notin \bar{E}_\alpha$ for all $\alpha < j(\kappa)$.
- Thus d extends every element of $\bigcup j" H = H$.
- Let $\hat{H}$ be $V[G * H * \hat{G}]$-generic for $j(\mathbb{Q})/d$.
- j lifts to $j : V[G * H] \to M[\hat{G} * \hat{H}]$.

I is a rigid precipitous ideal on $[\lambda]^\kappa = ([\lambda]^{\omega_1})^{V[G*H]}$:

$$I = \{X \in P([\lambda]^{\omega_1})^{V[G*H]} : \Vdash_{j(\mathbb{P})/i[G*H]*j(\mathbb{Q})/d} ([\mathrm{id}]_{\dot{U}} \notin j(X))\}.$$

- Since $M^{j(\kappa)} \cap V \subseteq M$, we have $(j(\kappa)$ is measurable$)^V$ and

$$\mathbb{R} = j(\mathbb{P}) * j(\mathbb{Q}) = [\mathrm{Col}(\omega, {<}j(\kappa)) * \mathrm{Code}(j(\kappa), j(\vec{E}))]^V$$

- We can view $V[G * H * \hat{G} * \hat{H}]$ as a forcing extension of V obtained by forcing with $\mathbb{R}$.
- By the lemma, in $V[G * H * \hat{G} * \hat{H}]$ there is a unique V-generic filter for $\mathbb{R}$.
- Thus, there is only one way to extend the embedding j to have domain $V[G * H]$.

- By Foreman's duality theorem, forcing over $V[G * H]$ with $j(\mathbb{P})/i[G * H] * j(\mathbb{Q})/d$ is equivalent to forcing over $V[G * H]$ with $P([\lambda]^{\omega_1})/I$ where

$$I = \{X \in P([\lambda]^{\omega_1})^{V[G*H]} : \Vdash_{j(\mathbb{P})/i[G*H]*j(\mathbb{Q})/d} ([\mathrm{id}]_{\dot{U}} \notin j(X))\}.$$

- Thus, forcing with $P([\lambda]^{\omega_1})/I$ over $V[G * H]$ produces an extension in which there is a unique generic for $P([\lambda]^{\omega_1})/I$.

□

A Rigid Presaturated Ideal on ω_1 + CH

Theorem (C.-Cox-Eskew, 2016)

Suppose κ is almost huge. Then there is a forcing extension in which there is a rigid presaturated ideal on ω_1 and GCH holds.

- Let $j : V \to M$ be an embedding with critical point κ and $M^{<j(\kappa)} \cap V \subseteq M$ obtained from an almost-huge tower.
- Let $G * (H \times K) \subseteq \mathbb{P} * \mathrm{Col}(\kappa, <j(\kappa)) * \mathrm{Code}(\kappa, \vec{E})$ be V-generic, so that $\kappa = \omega_1$.
- Generically lift j to $V[G * (H \times K)]$.
- The lift is the ultrapower by

$$U = \{X \in P(\kappa)^{V[G*(H\times K)]} : \kappa \in j(X)\}.$$

- $I = \{X \in P(\kappa)^{V[G*(H\times K)]} : \Vdash \kappa \notin j_{\dot{U}}(X)\}$ is a rigid presaturated ideal on ω_1.

Question

Is it consistent to have a saturated ideal on $\omega_1 + \mathrm{CH}$*?*