Force to change large cardinal strength

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Killing them softly

Suppose $\kappa \in V$ is a cardinal with large cardinal property A. The main idea is to find a notion of forcing $\mathbb{P}$ such that if $G \subseteq \mathbb{P}$ is V-generic, the cardinal κ no longer has property A in $V[G]$, but has as many of its other large cardinal properties as possible. The main theorems show how to do this for a class of cardinals.

Large cardinal degrees

This theme also includes degrees of various large cardinals:

t-inaccessible degrees
t-Mahlo degrees
Mitchell rank for measurable cardinals
Mitchell rank for supercompact cardinals

Degrees of measurable cardinals

The Mitchell relation is defined on measures μ and ν.

$$\mu \lhd \nu \iff \mu \in M_\nu$$

where M_ν is the ultrapower by ν.

Well-founded! See that $\mu \lhd \nu$ implies $j_\mu(\kappa) < j_\nu(\kappa)$.

Degrees of measurable cardinals

Let $m(\kappa)$ be the collection of normal measures on κ.

$o(\kappa)$ is the height of the Mitchell relation on κ.

$o(\kappa) = 0 \iff \kappa$ is not measurable
$o(\kappa) \geq 1 \iff \kappa$ is measurable
$o(\kappa) \geq 2 \iff$ there is a normal measure on κ which concentrates on measurable cardinals

If κ is a measurable cardinal, and μ is a normal measure on κ, and $j_\mu : V \to M_\mu$ is the ultrapower embedding by μ with critical point κ, then $o(\mu) = o(\kappa)^{M_\mu}$. Thus, $\beta < o(\kappa)$ if and only if there exists $j : V \to M$ elementary embedding with critical point κ with $o(\kappa)^M = \beta$.

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Lemma: Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties. If $\kappa > \delta$, then $o(\kappa)^{\bar{V}} \leq o(\kappa)^{V}$.

Sketch: This is a generalization of Corollary 22 of Hamkins 4. Assume the theorem holds for $\kappa' < \kappa$. If $o(\kappa)^{\bar{V}} > o(\kappa)^{V}$ get $j : \bar{V} \to \bar{M}$ with $o(\kappa)^{\bar{M}} = o(\kappa)^{V}$. By Hamkins 4 j is a lift of $j \restriction V : V \to M$ for some M. Then $o(\kappa)^{M} < o(\kappa)^{V} = o(\kappa)^{\bar{M}}$. This contradicts that j(IH) gives $o(\kappa)^{\bar{M}} \leq o(\kappa)^{M}$.

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Theorem: For any $V \models \mathrm{ZFC} + \mathrm{GCH}$, and any ordinal α, there is a forcing extension $V[G]$ where every cardinal $\kappa > \alpha$ has $o(\kappa)^{V[G]} = \min\{\alpha, o(\kappa)^V\}$.

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OUTLINE OF PROOF (by Induction):

$\mathbb{P}$ adds clubs which avoid high ranking cardinals.

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Mitchell rank cannot go up so $o(\kappa)^{V[G]} \leq o(\kappa)^V$.
Thus $o(\kappa)^{V[G]} = \min\{\alpha, o(\kappa)^V\}$ for all $\kappa < \alpha$.

Proof: Let $\alpha \in$ORD. Let $\mathbb{P}$ be an Easton support Ord-length iteration forcing at inaccessible stages γ to add a club $C_\gamma \subseteq \gamma$ such that $\delta \in C_\gamma$ implies $o(\delta)^V < \alpha$.

Let δ_0 denote the first inaccessible cardinal. $\mathbb{P}$ has a closure point at δ_0. Let $G \subseteq \mathbb{P}$ be V-generic. $V \subseteq V[G]$ has δ_0^+ approximation and cover properties. By the Lemma, Mitchell rank does not go up between V and $V[G]$.

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Proof continued:

Let $\kappa > \alpha$ be measurable. Fix $\beta_0 < \min\{\alpha, o(\kappa)^V\}$.

IH: $\forall \kappa' < \kappa$, $\kappa' > \alpha$, $o(\kappa')^{V[G]} = \min\{\alpha, o(\kappa')^V\}$.

Fix $j : V \to M, cp(j) = \kappa$, and $M \models o(\kappa) = \beta_0$. Lift j through $\mathbb{P}$ as follows. Forcing above κ is closed up to the next inaccessible. Call $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ the forcing up to and including stage κ. Let $G_\kappa \subseteq \mathbb{P}_\kappa$ be V-generic and $g \subseteq \mathbb{Q}$ be $V[G_\kappa]$-generic. First, need an M-generic filter for $j(\mathbb{P}_\kappa)$. Since $\alpha < \kappa$ and $cp(j) = \kappa$, the κth stage of $j(\mathbb{P}_\kappa)$ is $\mathbb{Q}$. $j(\mathbb{P}_\kappa) \equiv \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$. Need an $M[G_\kappa][g]$-generic filter for $\mathbb{P}_{\text{tail}}$ which is in $V[G_\kappa][g]$.

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Proof continued:

Diagonalize to get an $M[G_\kappa][g]$-generic filter for $\mathbb{P}_{\text{tail}}$:

$|\mathbb{P}_\kappa| = |\mathbb{Q}| = \kappa \Rightarrow \mathbb{P}_\kappa * \dot{\mathbb{Q}}$ has κ^+-chain condition

$M^\kappa \subseteq M \Rightarrow M[G_\kappa][g]^\kappa \subseteq M[G_\kappa][g]$

$2^\kappa = \kappa^+ \Rightarrow \mathbb{P}_\kappa$ has κ^+ many dense sets in V
$\Rightarrow \mathbb{P}_{\text{tail}}$ has at most $|j(\kappa^+)|^V \leq {\kappa^+}^\kappa = \kappa^+$ many dense sets in $M[G_\kappa][g]$

$\forall \beta < \kappa, \mathbb{P}_\kappa$ has a dense subset which is $\leq\beta$-closed
$\Rightarrow \mathbb{P}_{\text{tail}}$ has a dense subset which is $\leq\kappa$-closed.

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Proof continued:

Thus can build $G^* \subseteq \mathbb{P}_{\text{tail}}$ a generic filter on $V[G_\kappa][g]$. Thus $j(G_\kappa) = G_\kappa * g * G^*$ is M-generic for $j(\mathbb{P}_\kappa)$. And $j''G_\kappa \subseteq G_\kappa * g * G^*$ so lift j to $j : V[G_\kappa] \to M[j(G_\kappa)]$. Next lift through $\mathbb{Q}$. The forcing $j(\mathbb{Q})$ adds a club to $j(\kappa)$ containing no $\delta < j(\kappa)$ with $o(\delta)^M \geq \alpha$. Since $G_\kappa * g * G^* \in V[G_\kappa][g]$ and $M^\kappa \subseteq M$ it follows [Hamkins1 (Theorem 53)] that $M[j(G)]^\kappa \subseteq M[j(G)]$. The forcing $j(\mathbb{Q})$ has a dense subset which is $\leq\kappa$-closed. Since $|\mathbb{Q}| = \kappa \implies j(\mathbb{Q})$ has at most κ^+ many dense subsets.

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Proof continued:

Let $c = \cup g$ which is club in κ. Consider $\bar{c} = c \cup \{\kappa\}$ a closed, bounded subset of $j(\kappa)$, which contains no $\delta < \kappa$ with $o(\delta)^V \geq \alpha$. And $\bar{c} \in M[j(G_\kappa)]$. Then since $M \models o(\kappa) < \alpha$, the Lemma gives $M[j(G_\kappa)] \models o(\kappa) < \alpha$. Thus $\bar{c}$ is a condition in $j(\mathbb{Q})$. Thus, diagonalize to get an $M[j(G_\kappa)]$-generic filter $g^* \subseteq j(\mathbb{Q})$ which contains the master condition $\bar{c}$. Therefore lift j to $j : V[G_\kappa][g] \to M[j(G_\kappa)][j(g)]$ where $j(g) = g^*$.

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Proof continued:

Now apply the fully lifted j to IH to get $o(\kappa)^{M[j(G)]} = \beta_0$. Thus, $o(\kappa)^{V[G]} > \beta_0$. Since $\beta_0 < \min\{\alpha, o(\kappa)^V\}$ was arbitrary, we deduce $o(\kappa)^{V[G]} \geq \min\{\alpha, o(\kappa)^V\}$. $\mathbb{P}$ adds a club C to κ which concentrates on cardinals $\delta < \kappa$ with $o(\delta)^{V[G]} < \alpha$. Every normal measure contains C. So, no normal measures on κ can concentrate on cardinals of Mitchell rank α. Thus $o(\kappa)^{V[G]} \leq \alpha$. By the Lemma, $o(\kappa)^{V[G]} \leq o(\kappa)^V$. Thus $o(\kappa)^{V[G]} \leq \min\{\alpha, o(\kappa)^V\}$. Thus $o(\kappa)^{V[G]} = \min\{\alpha, o(\kappa)^V\} \forall \kappa > \alpha$. QED

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Thus $o(\kappa)^{V[G]} = \min\{\alpha, o(\kappa)^V\}$ for all $\kappa < \alpha$.

Representing Functions

Representing Functions:

The ordinal α is *represented* by $f\colon\kappa \to V_\kappa$ with respect to elementary embeddings $j : V \to M$ with critical point κ when $j(f)(\kappa) = \alpha$ for any such embedding j. Equivalently, consider only extender embeddings generated by seets up to $\max(\kappa, \alpha)$ to make the definition first order.

If there is such a function, α is *representable*.

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Theorem: Assume $F : Ord \to Ord$. Then there is a forcing extension $V[G]$ where, if κ is measurable and $F \restriction \kappa$ represents $F(\kappa)$, e.g. $F(\gamma) = \gamma$, then $o(\kappa) = \min\{o(\kappa)^V, F(\kappa)\}$.

Degrees of supercompact cardinals

Mitchell rank for θ-supercompactness:

κ is θ-supercompact μ, ν normal fine measures on $P_\kappa\theta$

$$\mu \lhd_{\theta\text{-sc}} \nu \iff \mu \in M_\nu$$

Killing supercompact cardinals softly

Lemma: Suppose $V \subseteq \bar{V}$ satisfies the δ approximation and cover properties. If $\kappa, \theta > \delta$, then $o_{\theta\text{-sc}}(\kappa)^{\bar{V}} \leq o_{\theta\text{-sc}}(\kappa)^{V}$.

Killing supercompact cardinals softly

Theorem: For any $V \models ZFC + GCH$, any $\Theta : \mathrm{Ord} \to \mathrm{Ord}$ and any $F : \mathrm{Ord} \to \mathrm{Ord}$, there is a forcing extension $V[G]$ where, if κ is $\Theta(\kappa)$-supercompact, $\Theta \restriction \kappa$ represents $\Theta(\kappa)$, $F \restriction \kappa$ represents $F(\kappa)$ in V, and $\Theta''\kappa \subseteq \kappa$, then $o_{\Theta(\kappa)\text{-sc}}(\kappa)^{V[g]} = \min\{o_{\Theta(\kappa)\text{-sc}}(\kappa)^V, F(\kappa)\}$.

Killing supercompact cardinals softly

The forcing: $\mathbb{P}$ is an Easton support Ord-length iteration which forces at γ to add a club $c_\gamma \subseteq \gamma$ such that $\delta \in c_\gamma$ implies $o_{\Theta(\delta)\text{-sc}}(\delta)^V < F(\delta)$, whenever γ is a closure point of Θ, meaning $\Theta''\gamma \subseteq \gamma$, and which otherwise forces trivially.

Questions

1. If $o(\kappa) = n$ is there a forcing extension where κ is least with $o(\kappa) = n$?

2. Is there a forcing extension where the least strongly compact cardinal is least with $o(\kappa) = 3$, the second strongly compact cardinal is least above this with $o(\kappa) = 2$ and the third strongly compact cardinal is the least measurable above this one?

3. Is there a forcing extension where the least strongly compact cardinal is the least measurable, the second strongly compact cardinal is least with $o(\kappa) = 2$, the third strongly compact cardinal is least with $o(\kappa) = 3$, ...

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Questions?

Thank you!

HAPPY BIRTHDAY JOEL!!!!!