Namba-like singularizations of successor cardinals

(Bukovský-)Namba forcing preserves $\kappa = \aleph_1$ and changes the cofinality of $\aleph_2$ to ω. We lift this to cardinals $\kappa > \aleph_1$. Assuming a measurable cardinal λ we construct models over which there is a further "Namba-like" forcing which preserves all cardinals $\leqslant \kappa$ and changes the cofinality of κ^+ to ω. Cofinalities different from ω can also be achieved by starting from measurable cardinals of sufficiently strong Mitchell order. Using core model theory one can show that the respective measurable cardinals are also necessary. This is joint work with Dominik Adolf (Münster).

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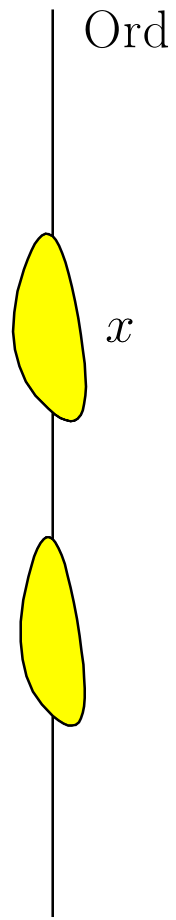
The Jensen Covering Theorem for L

Assume that $0^{\#}$ does not exist.

Then for every $x \subseteq \mathrm{Ord}$ there exists $z \in L$ such that

$$x \subseteq z \text{ and } \mathrm{card}(z) \leqslant \mathrm{card}(x) + \aleph_1$$

Where does $\aleph_1$ come from?



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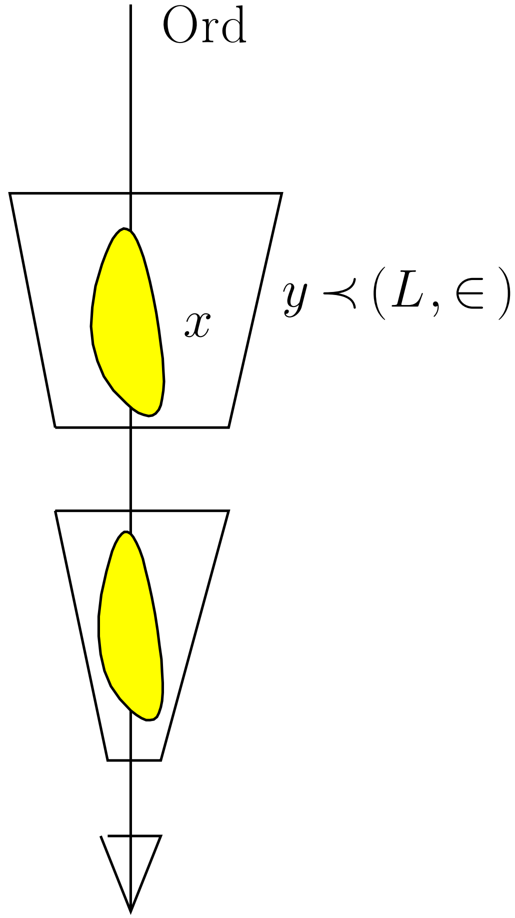

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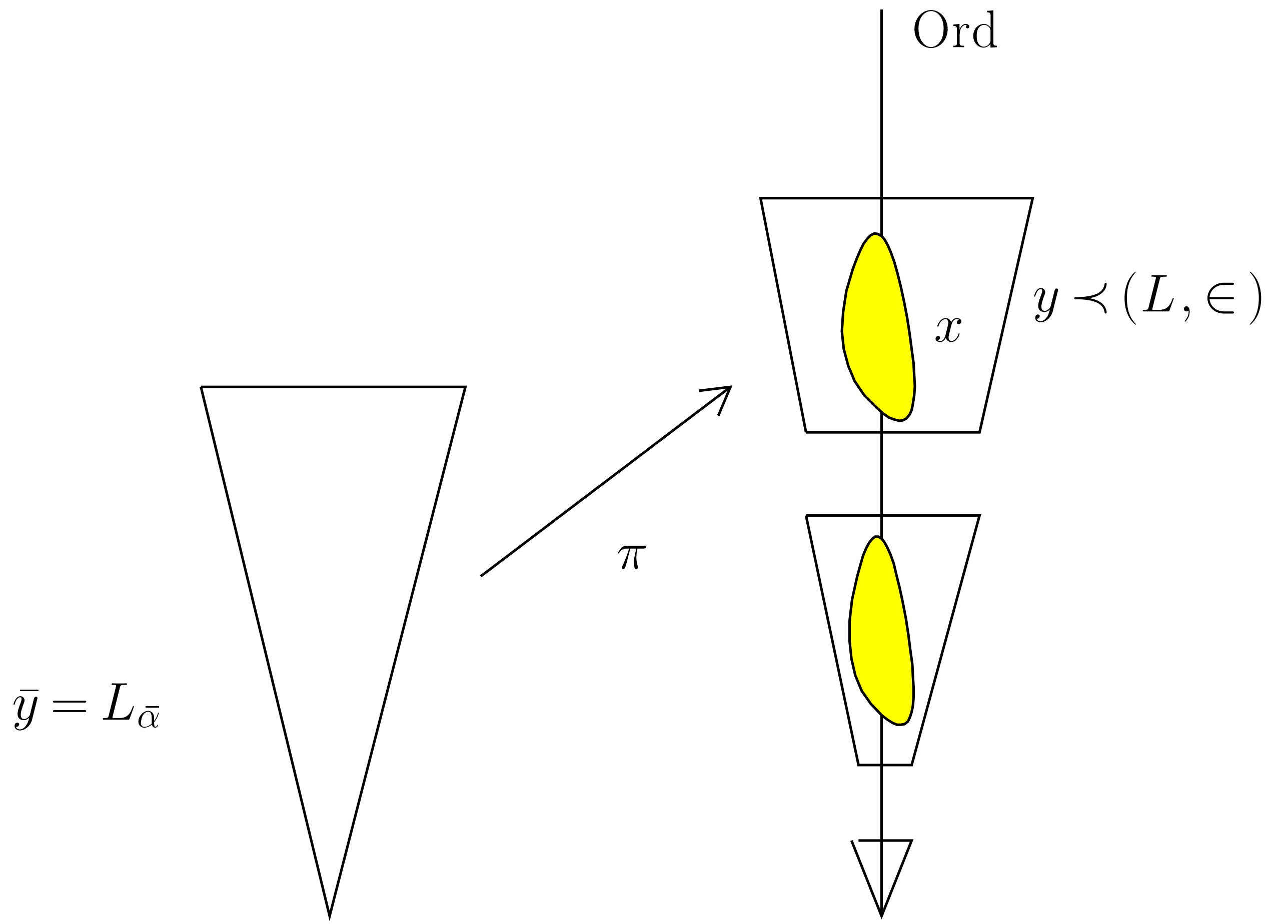

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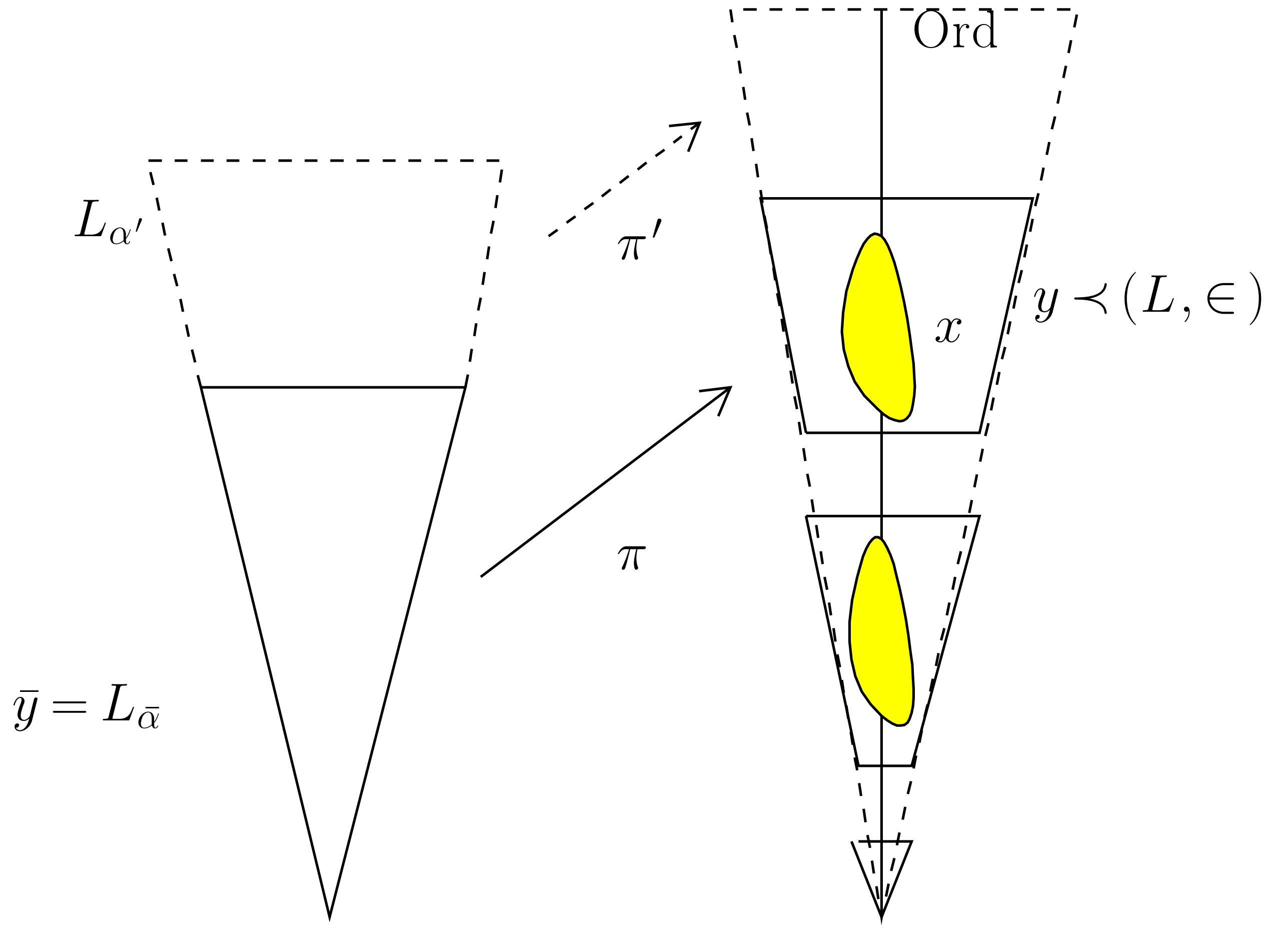

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- π can be extended to a $\pi' : L \to L$, and then $0^\#$ exists
- π can only be extended to a maximal $\pi' : L_{\alpha'} \to L_{\alpha''}$, and then one can define a covering set $z \supseteq x$ over $L_{\alpha''}$

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Problem: to ensure that the codomains to π' are wellfounded, Jensen includes $\aleph_1$ many possible counterexamples to wellfoundedness (vicious sequences) into the elementary substructure y, $x \subseteq y \prec L$.

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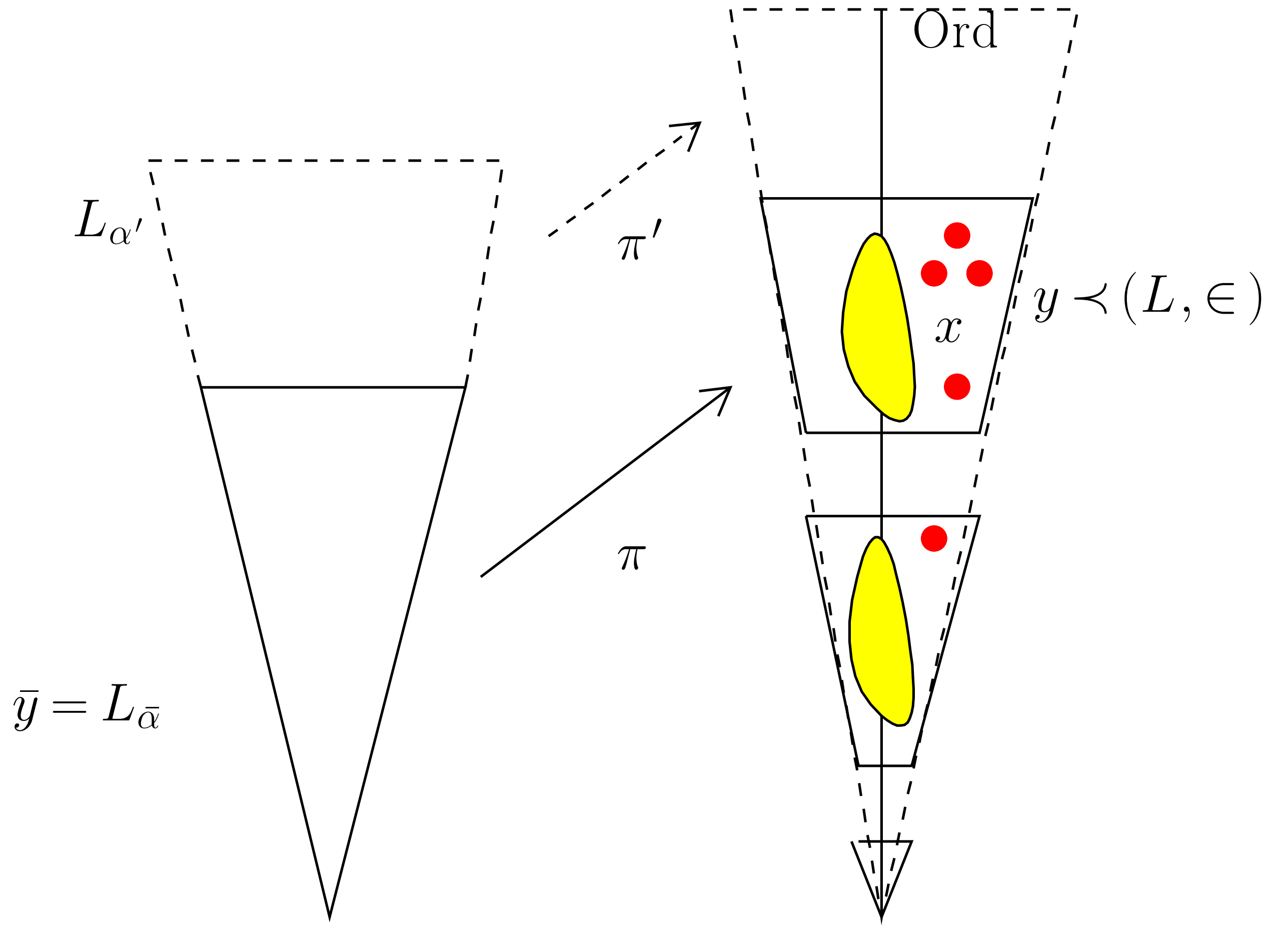

Namba forcing

(Assuming the continuum hypothesis) there is a forcing P_{Namba} such that

- $\mathcal{P}(\aleph_0)^{V[G_{\mathrm{Namba}}]} = \mathcal{P}(\aleph_0)^V$
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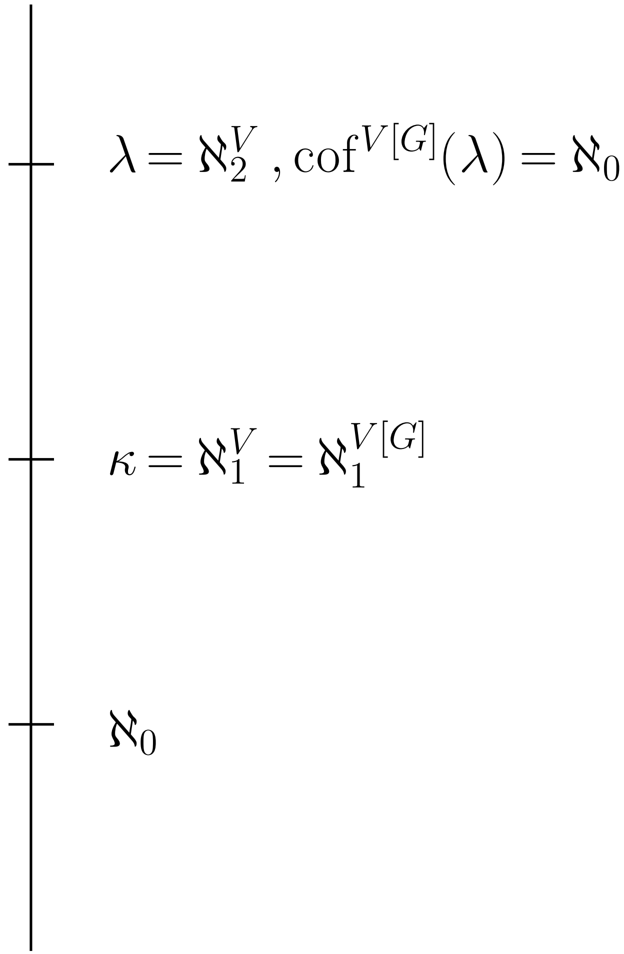

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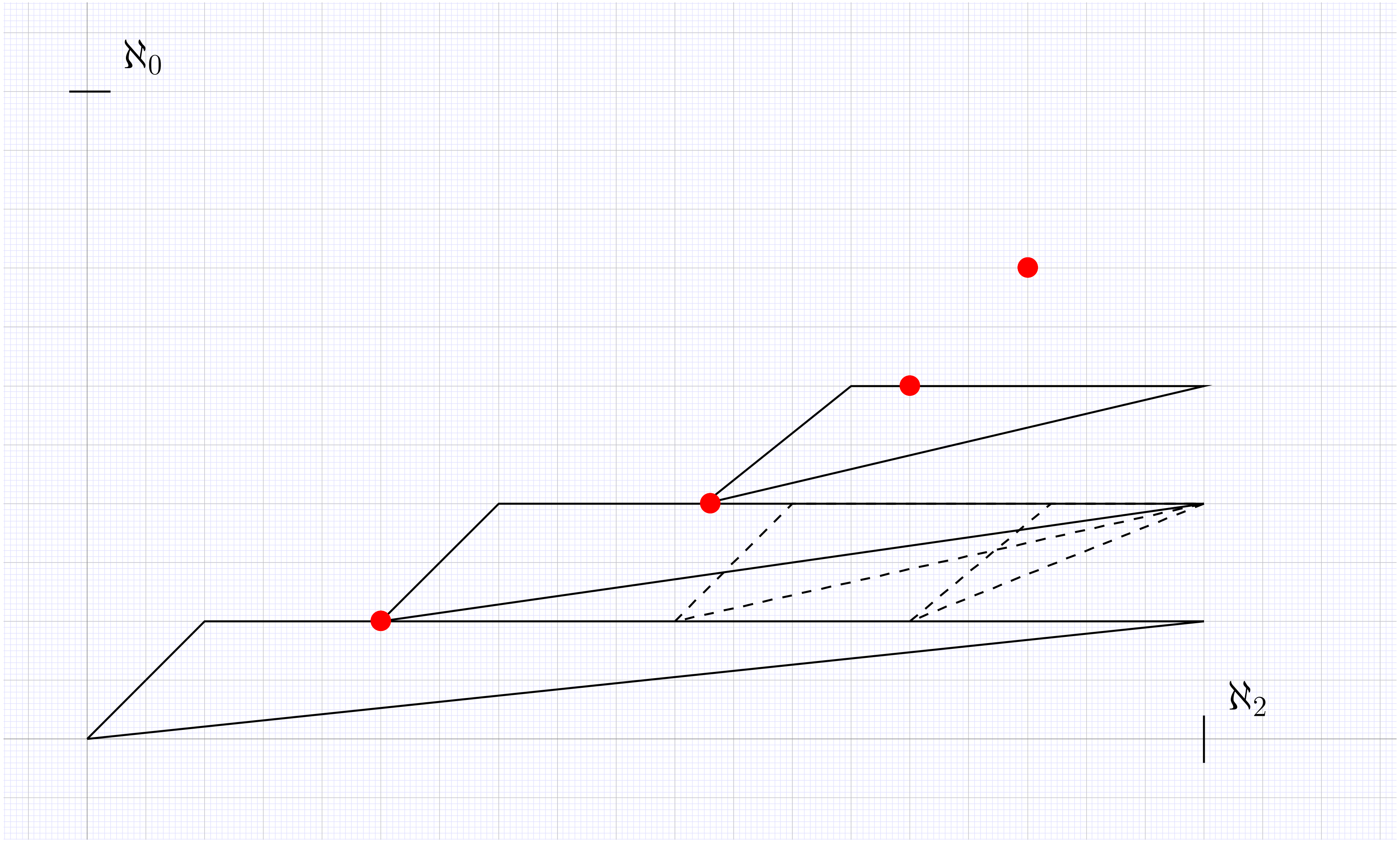

Namba forcing necessitates $\aleph_1$ in the Covering Theorem

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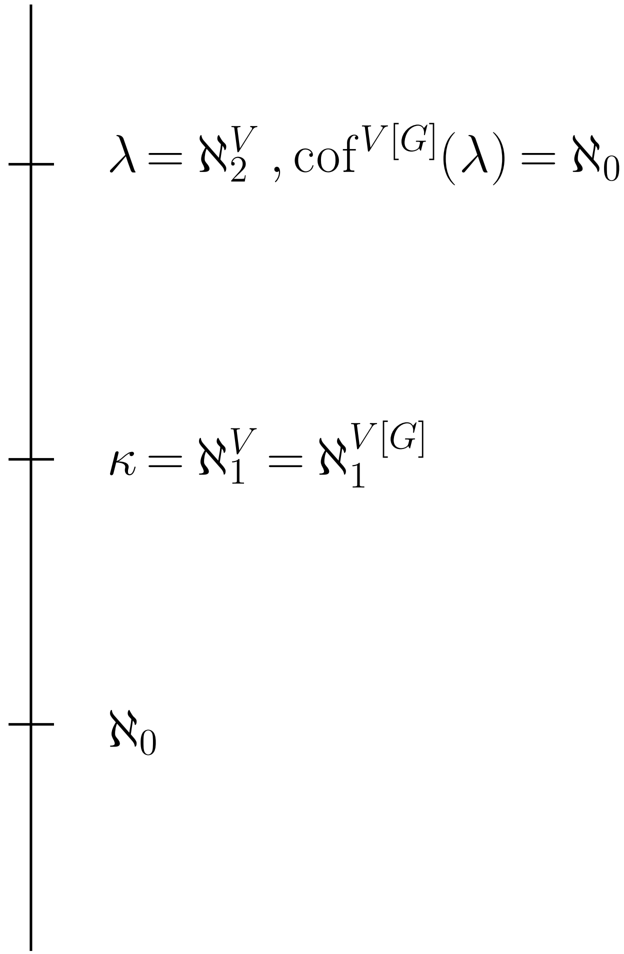
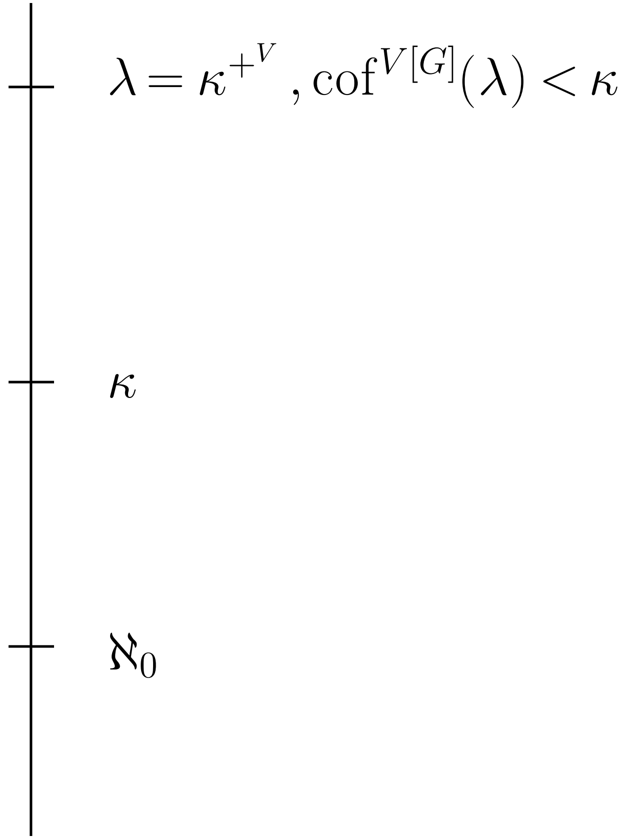


Can Namba forcing be generalized to cardinals κ bigger than $\aleph_1$?

For uncountable regular κ let NB_κ denote the property: there is a forcing P such that:

- $\mathcal{P}(<\kappa)^{V[G]} = \mathcal{P}(<\kappa)^V$
- κ is regular in $V[G]$
- $\mathrm{cof}^{V[G]}(\kappa^{+^V}) < \kappa$



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Contradiction

Hence Covering by L does not hold and $0^{\#}$ exists

The Dodd-Jensen Covering Theorem for the core model K_{DJ}

Assume that there is no inner model with a measurable cardinal.

Then for every $x \subseteq \mathrm{Ord}$ there exists $z \in K_{\mathrm{DJ}}$ such that

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The Dodd-Jensen Covering Theorem for the core model K_{DJ} . Thus NB_{κ} implies the existence of an inner model with a measurable cardinal

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Forcing NB_{κ} from a measurable cardinal

Let λ be a measurable cardinal

Use Prikry forcing to make $\mathrm{cof}(\lambda) = \aleph_0$

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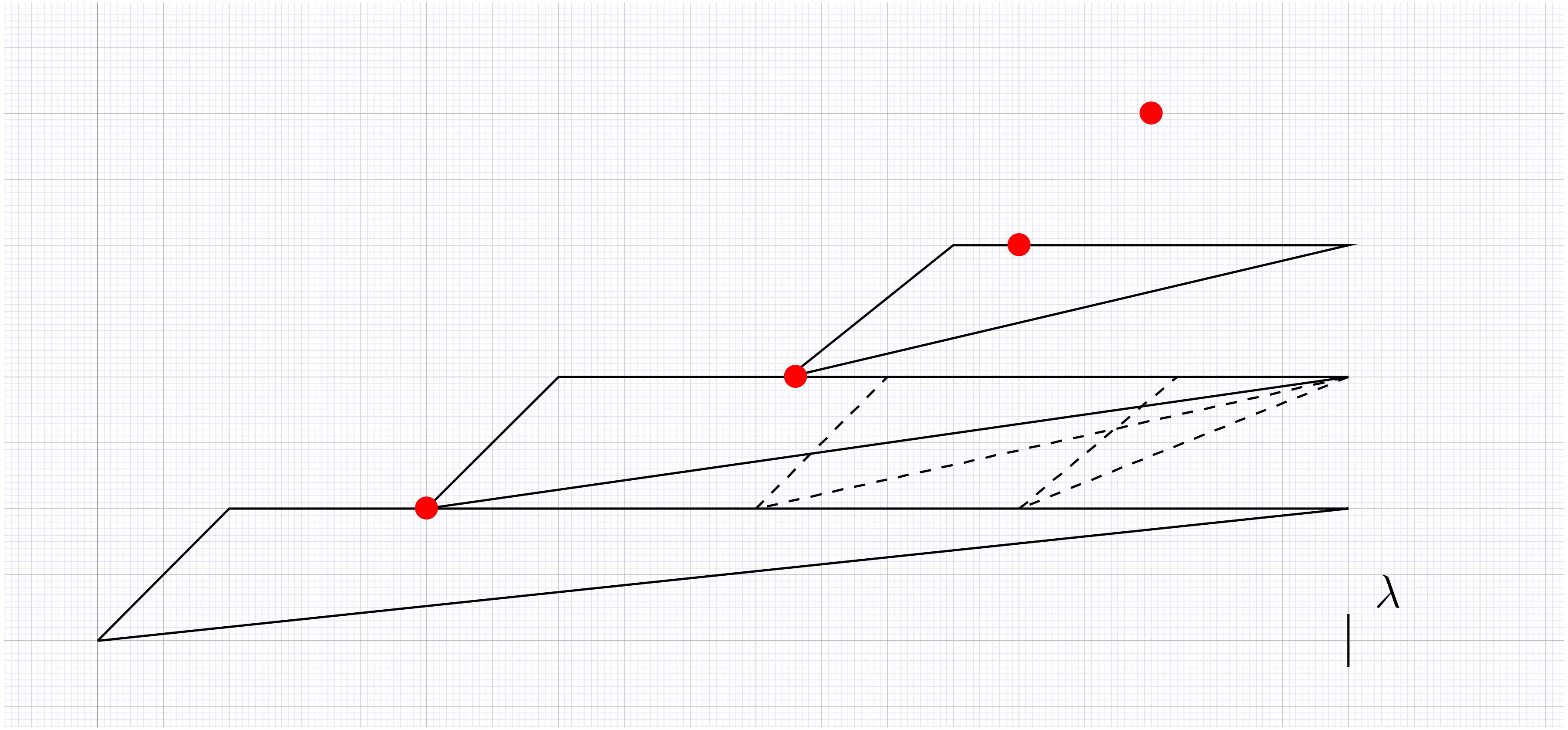
Problems

Prikry forcing singularizes λ so that λ is not inaccessible for Levy forcing

Levy forcing destroys the measurability of λ so that Prikry forcing does not work

Forcing NB_κ from a measurable cardinal

Prikry forcing



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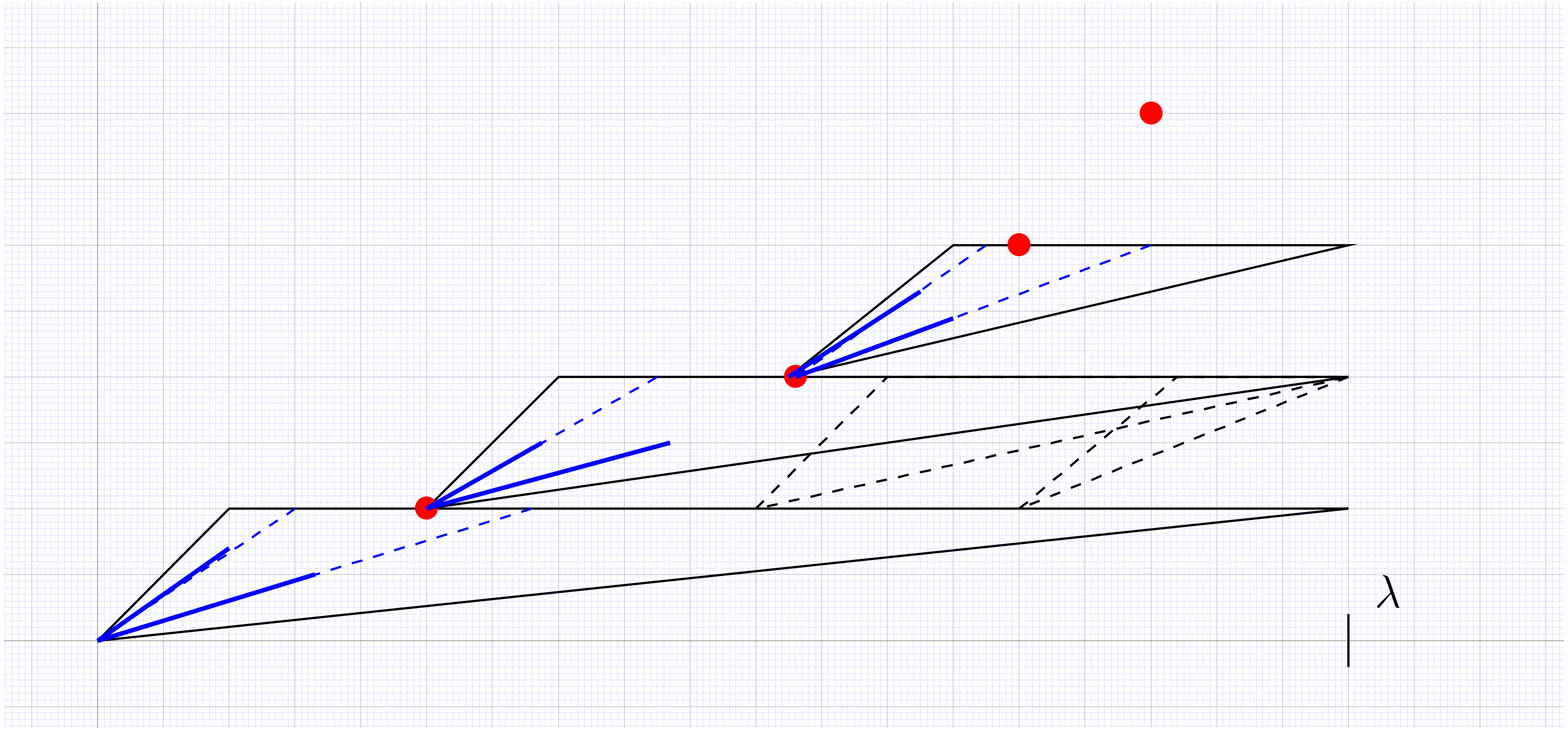
Define Q: Prikry forcing, combined with Levy collapses on the tree



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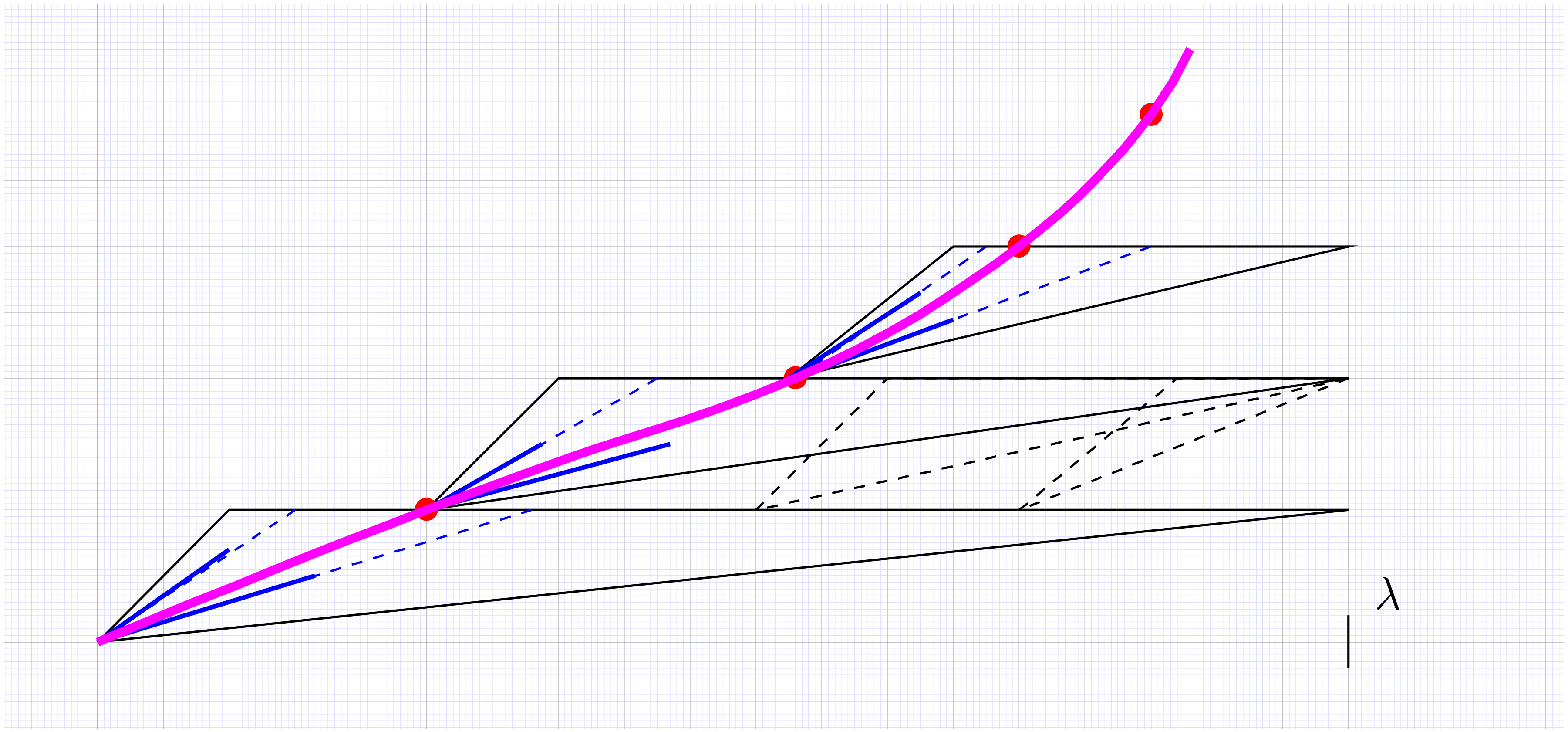

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Let G_Q be V-generic for Q

Then

- $\mathcal{P}(<\kappa)^{V[G_Q]} = \mathcal{P}(<\kappa)^V$
- κ is regular in $V[G_Q]$
- $\lambda = \kappa^{+^{V[H]}}$
- $\mathrm{cof}^{V[G_Q]}(\lambda) = \aleph_0$
- $V[G_Q] = V[H][G_Q/H]$

A general factorization result

Let R be a forcing that does not add bounded subsets of an inaccessible λ (like Prikry forcing for λ). Let G_R be V-generic for R. Let H be $V[G_R]$-generic for $\mathrm{Col}(\kappa, <\lambda)^{V[G_R]}$.

Then

$$\bar{H} = H \cap \mathrm{Col}(\kappa, <\lambda)^V = H \cap V$$

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Hence

$$V[G_R][H] = V[\bar{H}][(G_R \times H)/\bar{H}]$$

Forcing NB_κ from a measurable cardinal

$V[G_{\mathrm{Prikry}}][H] = V[\bar{H}][(G_{\mathrm{Prikry}} \times H)/\bar{H}]$ and hence

- $\mathcal{P}(<\kappa)^{V[G_{\mathrm{Prikry}}][H]} = \mathcal{P}(<\kappa)^V$
- κ is regular in $V[G_{\mathrm{Prikry}}][H]$
- $\lambda = \kappa^{+^{V[\bar{H}]}}$
- $\mathrm{cof}^{V[G_{\mathrm{Prikry}}][H]}(\lambda) = \aleph_0$

Hence $V[\bar{H}]$ satisfies NB_κ with $\mathrm{Prikry} * \mathrm{Col}(\kappa, <\lambda)/G_{\mathrm{Prikry}} \times H$

The V-genericity of $\bar{H}$

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Let $D = \{p \in \mathrm{Col}(\kappa, <\lambda)^{V[G_R]} \mid p \cap V_\mu \in \bar{D}\} \in V[G_R]$

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By $V[G_R]$-genericity take $p \in H \cap D$.

Then $p \cap V_\lambda \in \bar{D} \cap \bar{H} \neq \emptyset$

Thank You!