Namba-like singularizations of successor cardinals

(Bukovský-)Namba forcing preserves $\kappa = \aleph_1$ and changes the cofinality of $\aleph_2$ to $\omega$. We lift this to cardinals $\kappa > \aleph_1$. Assuming a measurable cardinal $\lambda$ we construct models over which there is a further "Namba-like" forcing which preserves all cardinals $\leq \kappa$ and changes the cofinality of $\kappa^+$ to $\omega$. Cofinalities different from $\omega$ can also be achieved by starting from measurable cardinals of sufficiently strong Mitchell order. Using core model theory one can show that the respective measurable cardinals are also necessary. This is joint work with Dominik Adolf (Münster).

Peter Koepke, Mathematical Institute, University of Bonn, Germany
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The Jensen Covering Theorem for $L$

Assume that $\emptyset^\# \text{ does not exist.}$

Then for every $x \subseteq \text{Ord}$ there exists $z \in L$ such that

$$x \subseteq z \quad \text{and} \quad \text{card}(z) \leq \text{card}(x) + \aleph_1$$
Where does $\aleph_1$ come from?
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$\bar{y} = L_{\bar{\alpha}}$

Ord

$y \prec (L, \in)$
Where does $\aleph_1$ come from?
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By ultrapower-like techniques, \( \pi \) can be extended to \( L_{\alpha'} \supseteq L_{\alpha} \).
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Dichotomy:

- $\pi$ can be extended to a $\pi': L \rightarrow L$, and then $0^#$ exists

- $\pi$ can only be extended to a maximal $\pi': L_{\alpha'} \rightarrow L_{\alpha''}$, and then one can define a covering set $z \supseteq x$ over $L_{\alpha''}$
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Dichotomy:

- $\pi$ can be extended to a $\pi': L \to L$, and then $0^\#$ exists

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Problem: to ensure that the codomains to $\pi'$ are wellfounded, Jensen includes $\aleph_1$ many possible counterexamples to wellfoundedness (vicious sequences) into the elementary substructure $y, x \subseteq y < L$. 
Where does \( \aleph_1 \) come from?

\[
\bar{y} = L_\alpha
\]
Namba forcing

(Assuming the continuum hypothesis) there is a forcing $P_{\text{Namba}}$ such that

1. $\mathcal{P}(\aleph_0)^V[G_{\text{Namba}}] = \mathcal{P}(\aleph_0)^V$

2. $\aleph_1^{V[G_{\text{Namba}}]} = \aleph_1^V$

3. $\text{cof}^V[G_{\text{Namba}}](\aleph_2^V) = \aleph_0$
Namba forcing

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- $\aleph_1^{V[G_{\text{Namba}}]} = \aleph_1^V$
- $\text{cof}^{V[G_{\text{Namba}}]}(\aleph_2^V) = \aleph_0$

$\lambda = \aleph_2^V$, $\text{cof}^{V[G]}(\lambda) = \aleph_0$

$\kappa = \aleph_1^V = \aleph_1^{V[G]}$

$\aleph_0$
Namba forcing
Namba forcing necessitates $\aleph_1$ in the Covering Theorem

Let $x \subseteq \lambda$ be cofinal, $\text{otp}(x) = \aleph_0$, $x \in L[G_{\text{Namba}}]$
Namba forcing necessitates $\aleph_1$ in the Covering Theorem

Let $x \subseteq \lambda$ be cofinal, $\text{otp}(x) = \aleph_0$, $x \in L[G_{\text{Namba}}]$

Assume that $x$ can be covered by a countable $z \in L : x \subseteq z \subseteq \text{Ord}$

Then $\text{otp}(z) < \aleph_1 < \lambda$
Namba forcing necessitates $\aleph_1$ in the Covering Theorem

Let $x \subseteq \lambda$ be cofinal, $\text{otp}(x) = \aleph_0$, $x \in L[G_{Namba}]$

Assume that $x$ can be covered by a countable $z \in L : x \subseteq z \subseteq \text{Ord}$

Then $\text{otp}(z) < \aleph_1 < \lambda$

cof$_{L}(\lambda) < \lambda$, although $\lambda$ is regular in $L$

Contradiction
Can Namba forcing be generalized to cardinals $\kappa$ bigger than $\aleph_1$?

Namba forcing $P_{\text{Namba}}$:

- $\mathcal{P}(\aleph_0)^{V[G_{\text{Namba}}]} = \mathcal{P}(\aleph_0)^V$

- $\aleph_1^{V[G_{\text{Namba}}]} = \aleph_1^V$

- $\text{cof}^{V[G_{\text{Namba}}]}(\aleph_2^V) = \aleph_0$

- $\lambda = \aleph_2^V$, $\text{cof}^{V[G]}(\lambda) = \aleph_0$

- $\kappa = \aleph_1^V = \aleph_1^{V[G]}$

- $\aleph_0$
Can Namba forcing be generalized to cardinals $\kappa$ bigger than $\aleph_1$?

For uncountable regular $\kappa$ let $NB_\kappa$ denote the property: there is a forcing $P$ such that:

- $P(\kappa)^{V[G]} = P(\kappa)^V$
- $\kappa$ is regular in $V[G]$
- $\text{cof}^{V[G]}(\kappa^+) < \kappa$

$\lambda = \kappa^+, \text{cof}^{V[G]}(\lambda) < \kappa$

$\kappa$

$\aleph_0$
NB$_\kappa$ implies the existence of $0^#$

Let $x \subseteq \lambda$ be cofinal, otp$(x) < \kappa$, $x \in V[G]$
NB$_\kappa$ implies the existence of $0^#$

Let $x \subseteq \lambda$ be cofinal, $\text{otp}(x) < \kappa$, $x \in V[G]$

Assume that $x$ can be covered by $z \in L$:

\[ x \subseteq z \subseteq \text{Ord} \quad \text{and} \quad \text{card}(z) \leq \text{card}(x) + \aleph_1 \]
\( \textbf{NB}_\kappa \) implies the existence of \( 0^# \)

Let \( x \subseteq \lambda \) be cofinal, \( \text{otp}(x) < \kappa \), \( x \in V[G] \)

Assume that \( x \) can be covered by \( z \in L \) :

\[
x \subseteq z \subseteq \text{Ord} \quad \text{and} \quad \text{card}(z) \leq \text{card}(x) + \aleph_1
\]

Then \( \text{otp}(z) < \kappa < \lambda \)
NB$_\kappa$ implies the existence of 0#

Let $x \subseteq \lambda$ be cofinal, otp$(x) < \kappa$, $x \in V[G]$

Assume that $x$ can be covered by $z \in L$:

$$x \subseteq z \subseteq \text{Ord and card}(z) \leq \text{card}(x) + \aleph_1$$

Then otp$(z) < \kappa < \lambda$

$\cof^L(\lambda) < \lambda$, although $\lambda$ is regular in $L$

Contradiction
NB$_\kappa$ implies the existence of 0#$^\#$

Let $x \subseteq \lambda$ be cofinal, $\text{otp}(x) < \kappa$, $x \in V[G]$

Assume that $x$ can be covered by $z \in L$:

$$x \subseteq z \subseteq \text{Ord} \quad \text{and} \quad \text{card}(z) \leq \text{card}(x) + \aleph_1$$

Then $\text{otp}(z) < \kappa < \lambda$

$\text{cof}^L(\lambda) < \lambda$, although $\lambda$ is regular in $L$

Contradiction

Hence Covering by $L$ does not hold and 0#$^\#$ exists
The Dodd-Jensen Covering Theorem for the core model $K_{DJ}$

Assume that there is no inner model with a measurable cardinal.

Then for every $x \subseteq \text{Ord}$ there exists $z \in K_{DJ}$ such that

$$x \subseteq z \text{ and } \text{card}(z) \leq \text{card}(x) + \aleph_1$$
The Dodd-Jensen Covering Theorem for the core model $K_{DJ}$. Thus $NB_\kappa$ implies the existence of an inner model with a measurable cardinal.

Assume that there is no inner model with a measurable cardinal.

Then for every $x \subseteq \text{Ord}$ there exists $z \in K_{DJ}$ such that

$$x \subseteq z \text{ and } \text{card}(z) \leq \text{card}(x) + \aleph_1$$
Forcing $\mathbf{NB}_\kappa$ from a measurable cardinal

Let $\lambda$ be a measurable cardinal

Use Prikry forcing to make $\text{cof}(\lambda) = \aleph_0$

Use Levy forcing to make $\lambda = \kappa^+$
Forcing $\text{NB}_\kappa$ from a measurable cardinal

Let $\lambda$ be a measurable cardinal

Use Prikry forcing to make $\text{cof}(\lambda) = \aleph_0$

Use Levy forcing to make $\lambda = \kappa^+$

Problems

Prikry forcing singularizes $\lambda$ so that $\lambda$ is not inaccessible for Levy forcing

Levy forcing destroys the measurability of $\lambda$ so that Prikry forcing does not work
Forcing $NB_\kappa$ from a measurable cardinal

Prikry forcing
Forcing $\text{NB}_\kappa$ from a measurable cardinal

Define $Q$: Prikry forcing, combined with Levy collapses on the tree
Forcing $\mathbf{NB}_\kappa$ from a measurable cardinal

Define $Q$: Prikry forcing, combined with $\text{Levy collapses}$ on the tree

Union $H$ of $\text{Levy collapses}$ along generic branch is $V$-generic for $\text{Col}(\kappa, < \lambda)$
Forcing $\text{NB}_\kappa$ from a measurable cardinal

Define $Q$: Prikry forcing, combined with Levy collapses on the tree

Union $H$ of Levy collapses along generic branch is $V$-generic for $\text{Col}(\kappa, < \lambda)$

Let $G_Q$ be $V$-generic for $Q$

Then

- $\mathcal{P}( < \kappa)^{V[G_Q]} = \mathcal{P}( < \kappa)^V$
- $\kappa$ is regular in $V[G_Q]$
- $\lambda = \kappa^+^{V[H]}$
- $\text{cof}^{V[G_Q]}(\lambda) = \aleph_0$
- $V[G_Q] = V[H][G_Q/H]$
A general factorization result

Let $R$ be a forcing that does not add bounded subsets of an inaccessible $\lambda$ (like Prikry forcing for $\lambda$). Let $G_R$ be $V$-generic for $R$. Let $H$ be $V[G_R]$-generic for $\text{Col}(\kappa, < \lambda)^{V[G_R]}$.

Then

$$\bar{H} = H \cap \text{Col}(\kappa, < \lambda)^V = H \cap V$$

is $V$-generic for $\text{Col}(\kappa, < \lambda)^V$
A general factorization result

Let $R$ be a forcing that does not add bounded subsets of an inaccessible $\lambda$ (like Prikry forcing for $\lambda$). Let $G_R$ be $V$-generic for $R$. Let $H$ be $V[G_R]$-generic for $\text{Col}(\kappa, < \lambda)^{V[G_R]}$.

Then

$$\bar{H} = H \cap \text{Col}(\kappa, < \lambda)^V = H \cap V$$

is $V$-generic for $\text{Col}(\kappa, < \lambda)^V$

Hence

$$V[G_R][H] = V[\bar{H}][(G_R \times H)/\bar{H}]$$
Forcing $\text{NB}_\kappa$ from a measurable cardinal

$V[G_{\text{Prikry}}][H] = V[\bar{H}][G_{\text{Prikry}} \times H]/\bar{H}$ and hence

- $\mathcal{P}(\kappa)^{V[G_{\text{Prikry}}][H]} = \mathcal{P}(\kappa)^V$

- $\kappa$ is regular in $V[G_{\text{Prikry}}][H]$

- $\lambda = \kappa^{+^{V[H]}}$

- $\text{cof}^{V[G_{\text{Prikry}}][H]}(\lambda) = \aleph_0$

Hence $V[\bar{H}]$ satisfies $\text{NB}_\kappa$ with $\text{Prikry} \ast \text{Col}(\kappa, < \lambda)/G_{\text{Prikry}} \times H$
The $V$-genericity of $\bar{H}$

Let $\bar{D} \in V$ be dense in $Col(\kappa, < \lambda)^V$. 
The $V$-genericity of $\bar{H}$

Let $\bar{D} \in V$ be dense in $\text{Col}(\kappa, < \lambda)^V$.

There is $\mu < \lambda$ with $\text{cof}^V(\mu) \geq \kappa$ such that $\bar{D} \cap V_\mu$ is dense in $\text{Col}(\kappa, < \lambda)^V \cap V_\mu$. 
The $V$-genericity of $\bar{H}$

Let $\bar{D} \in V$ be dense in $\text{Col}(\kappa, < \lambda)^V$.

There is $\mu < \lambda$ with $\text{cof}^V(\mu) \geq \kappa$ such that $\bar{D} \cap V_\mu$ is dense in $\text{Col}(\kappa, < \lambda)^V \cap V_\mu$.

Let $D = \{ p \in \text{Col}(\kappa, < \lambda)^{V[G_R]} | p \cap V_\mu \in \bar{D} \} \in V[G_R]$.

$D$ is dense in $\text{Col}(\kappa, < \lambda)^{V[G_R]}$. 
The \( V \)-genericity of \( \vec{H} \)

Let \( \vec{D} \in V \) be dense in \( \text{Col}(\kappa, < \lambda)^V \).

There is \( \mu < \lambda \) with \( \text{cof}^V(\mu) \geq \kappa \) such that \( \vec{D} \cap V_\mu \) is dense in \( \text{Col}(\kappa, < \lambda)^V \cap V_\mu \).

Let \( D = \{ p \in \text{Col}(\kappa, < \lambda)^{V[G_R]} \mid p \cap V_\mu \in \vec{D} \} \in V[G_R] \)

\( D \) is dense in \( \text{Col}(\kappa, < \lambda)^{V[G_R]} \).

By \( V[G_R] \)-genericity take \( p \in H \cap D \).

Then \( p \cap V_\lambda \in \vec{D} \cap \vec{H} \neq \emptyset \)
Thank You!