# Weak Compactness without Inaccessibility

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*Set Theory Day, celebrating*
*Joel David Hamkins' 50th birthday*

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# Acknowledgements

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Happy birthday, Joel!

## Theorem (folklore)

*A cardinal $\kappa$ is weakly compact iff*

1. *$\kappa$ is inaccessible, and*
2. *(embedding property): for every transitive set $M$ of size $\kappa$ with $\kappa \in M$ there is a transitive set $N$ and an elementary embedding $j : M \to N$ with critical point $\kappa$.*

We denote the embedding property in 2 by $\mathrm{EP}(\kappa)$. Thus:

- $\mathrm{EP}(\kappa)$ implies $\kappa$ is regular cardinal:
  *Proof:* If $\kappa$ is singular, and $f : \lambda \to \kappa$ is cofinal with $\lambda < \kappa$, then put $f$ into such transitive set $M$, and find $j : M \to N$. Then $j(f) : \lambda \to j(\kappa)$ is cofinal, yet $j(f) = f$, a contradiction. QED.
- $\mathrm{EP}(\kappa)$ implies $\kappa$ is limit cardinal:
  *Proof:* If $\kappa = \delta^+$, then find $M$ which sees this and $j : M \to N$, and so $j(\kappa) = \delta^+$ in $N$, a contradiction. QED.
- $\mathrm{EP}(\kappa)$ implies $\kappa$ is uncountable.
- Thus: $\mathrm{EP}(\kappa)$ **implies $\kappa$ is weakly inaccessible**

$\mathrm{EP}(\kappa)$: For every transitive set $M$ of size $\kappa$ with $\kappa \in M$ there is a transitive set $N$ and an elementary embedding $j : M \to N$ with critical point $\kappa$.

**More easy consequences of $\mathrm{EP}(\kappa)$**

- $\kappa$ is weakly Mahlo:
  *Proof:* Let $S = \{\alpha < \kappa : \alpha \text{ is regular}\}$, and let $C \subseteq \kappa$ be any club. Find such $M$ with $M \prec H_{\kappa^+}$ and $C \in M$. If $j : M \to N$, then $\kappa \in j(C)$ is regular, and so $S \cap C \neq \emptyset$, as desired. QED.
- $\kappa$ has the tree property, i.e. every $\kappa$-tree has a cofinal branch:
  *Proof:* Assume w.l.o.g. $T \subseteq \kappa$. Put $T$ into such $M$ with $M \prec H_{\kappa^+}$, and let $j : M \to N$. Then $j(T)$ is tree of height $j(\kappa)$ that agrees with $T$ on the levels below $\kappa$. Thus if $t \in j(T)$ is any node in the $\kappa^{\text{th}}$ levels of $T$, then the predecessors of $t$ form a $\kappa$-branch through $T$. QED.
- $\kappa \neq 2^\delta$ for any cardinal $\delta$.
  *Proof:* Otherwise, let $\delta < \kappa$ with $2^\delta = \kappa$. If $f : \kappa \to P(\delta)$ is bijection in such $M$ with $M \prec H_{\kappa^+}$ and $j : M \to N$, then $j(f) : j(\kappa) \to P(\delta)$ is bijection, but $j(f) \restriction \kappa = f$, so the range of $j(f) \restriction \kappa$ already exhausts all of $P(\delta)$, a contradiction. QED.
- $\kappa$ is weakly compact in $L$. (not difficult to show)

Thus: if $\mathrm{EP}(\kappa)$ with $2^{<\kappa} = \kappa$, then $\kappa$ is inaccessible, and hence weakly compact.

Thus: If we want to have $\mathrm{EP}(\kappa)$ without inaccessible $\kappa$, then we could try to blow up powerset $P(\delta)$ for some $\delta < \kappa$. Indeed:

## Observation (Hamkins)

*It is relatively consistent with* $\mathrm{ZFC}$ *that there is a cardinal* $\kappa < 2^\omega$ *with the embedding property* $\mathrm{EP}(\kappa)$*.*
*Indeed, if* $\kappa$ *is weakly compact, then after forcing to add* $\kappa^+$ *many Cohen reals, then cardinal* $\kappa < 2^\omega$*, yet the embedding property* $\mathrm{EP}(\kappa)$ *holds.*

## Proof.

A simple lifting argument.
Let $G \subseteq \mathrm{Add}(\omega, \kappa^+)$ be $V$-generic. We shall show that $\mathrm{EP}(\kappa)$ holds in $V[G]$. Fix any transitive set $M_0$ of size $\kappa$ with $\kappa \in M_0$. Code $M_0$ by a subset $A \subseteq \kappa$ in $V[G]$, with nice name $\dot{A} \in V$. Since forcing is c.c.c, may assume without loss by automorphism that $\dot{A}$ is an $\mathrm{Add}(\omega, \kappa)$-name, so $A = \dot{A}_{G \restriction \kappa}$. Place $\dot{A} \in M$ with $M \prec H_{\kappa^+}$ and get $j : M \to N$ with critical point $\kappa$, and $|N| = \kappa$. As $j(\kappa) < \kappa^+$, the filter $G \restriction j(\kappa)$ is $N$-generic for $\mathrm{Add}(\omega, j(\kappa))$, and so embedding lifts to $j : M[G \restriction \kappa] \to N[G \restriction j(\kappa)]$. Since $\dot{A} \in M$, it follows that $A \in M[G \restriction \kappa]$, and thus $M_0 \in M[G \restriction \kappa]$, and $j \restriction M_0 : M_0 \to j(M_0)$ is the desired embedding. □

**Goal:** Investigate how $\mathrm{EP}(\kappa)$ relates to some of the other characterizations of weak compactness; particular interest, when $2^\delta > \kappa$ for some $\delta < \kappa$; aim to find equivalent formulations.

## Theorem (folklore – characterizations of weakly compact cardinals)

*Assume that $\kappa$ is inaccessible. Then the following are equivalent:*

1. *(embedding property) For every transitive set $M$ of size $\kappa$ with $\kappa \in M$, there is a transitive set $N$ and an elementary embedding $j : M \to N$ with critical point $\kappa$.*
2. *(extension property) For every $A \subseteq V_\kappa$, there is a transitive set $W'$ and $A' \subseteq W'$ such that $\langle V_\kappa, \in, A\rangle \prec \langle W', \in, A'\rangle$.*
3. *(weak compactness property) Every $\kappa$-satisfiable theory of size $\kappa$ in an $L_{\kappa,\kappa}$-language is satisfiable.*
4. *($\Pi^1_1$ indescribability property) For every $A \subseteq V_\kappa$ and every $\Pi^1_1$ sentence $\varphi$ such that $\langle V_\kappa, \in, A\rangle \models \varphi$, there is $\delta < \kappa$ such that $\langle V_\delta, \in, A \cap V_\delta\rangle \models \varphi$.*
5. *(filter property) For every collection $\mathcal{A}$ of at most $\kappa$ many subsets of $\kappa$, there is a $\kappa$-complete nonprincipal filter measuring every set in $\mathcal{A}$.*
6. *(partition property) $\kappa$ has the partition property $\kappa \to (\kappa)^2_2$.*
7. *(tree property) $\kappa$ has the tree property*

## Theorem (folklore; embedding iff extension, for inaccessible $\kappa$)

*Assume that $\kappa$ is inaccessible. Then the following are equivalent.*

1. *(embedding property) For every transitive set $M$ of size $\kappa$ with $\kappa \in M$, there is transitive set $N$ and elementary embedding $j : M \to N$ with critical point $\kappa$*
2. *(extension property) For every $A \subseteq V_\kappa$, there is a transitive set $W'$ and $A' \subseteq W'$ such that $\langle V_\kappa, \in, A\rangle \prec \langle W', \in, A'\rangle$.*

## Proof.

1→2: Fix any $A \subseteq V_\kappa$. Find any transitive $M$ of size $\kappa$ with $V_\kappa, A, \kappa \in M$ and let $j : M \to N$ with critical point $\kappa$. Then $j \restriction V_\kappa : \langle V_\kappa, \in, A\rangle \to \langle j(V_\kappa), \in, j(A)\rangle$ is elementary, and since $j \restriction V_\kappa = \mathrm{id}$, it follows that $\langle V_\kappa, \in, A\rangle \prec \langle j(V_\kappa), \in, j(A)\rangle$, as desired.

2→1: Fix any such $M$, and let $E$ be well-founded relation so that $\pi : \langle \kappa, E\rangle \to \langle M, \in\rangle$ is isomorphism. By 2, there is transitive $W'$ and $E' \subseteq W'$ such that $\langle V_\kappa, \in, E\rangle \prec \langle W', \in, E'\rangle$. Let $\gamma = \mathrm{ORD}^{W'}$. Using the club $C$ of all $\alpha < \kappa$ such that $\langle V_\alpha, \in, E \cap V_\alpha\rangle \prec \langle V_\kappa, \in, E\rangle$ as additional predicate, if necessary, assume without loss that $\mathrm{cof}(\gamma) > \omega$, and consequently that $E'$ is well-founded on $\gamma$. If $\pi^* : \langle \gamma, E'\rangle \to \langle N, \in\rangle$ is the Mostowski collapse, then the map $\pi^* \circ \pi^{-1}$ is an elementary embedding from $M$ to $N$ with critical point $\kappa$. □

## Theorem (embedding $\mathrm{EP}(\kappa)$ iff extension $\mathrm{XP}(\kappa)$)

*Assume that $\kappa$ is any cardinal. Then the following are equivalent.*

1. $\mathrm{EP}(\kappa)$: *For every transitive set $M$ of size $\kappa$ with $\kappa \in M$, there is transitive $N$ and elementary $j : M \to N$ with critical point $\kappa$*
2. *extension property* $\mathrm{XP}(\kappa)$: *For every transitive $W \subseteq H_\kappa$ of size $\kappa$ and $A \subseteq W$ there is transitive set $W'$ and $A' \subseteq W'$ such that $\langle W, \in, A\rangle \prec \langle W', \in A'\rangle$.*

## Proof.

1→2: Fix any transitive $W \subseteq H_\kappa$ of size $\kappa$ and any $A \subseteq W$. Find transitive $M$ of size $\kappa$ with $W, A, \kappa \in M$ and let $j : M \to N$ with critical point $\kappa$. Then $j \restriction W : \langle W, \in, A\rangle \to \langle j(W), \in, j(A)\rangle$ is elementary, and since $W \subseteq H_\kappa$, it follows that $j \restriction W = \mathrm{id}$, and so $\langle V_\kappa, \in, A\rangle \prec \langle j(V_\kappa), \in, j(A)\rangle$, as desired.
2→1: Fix any such $M$, and let $E \subseteq \kappa \times \kappa$ code $M$ so that $\pi : \langle \kappa, E\rangle \to \langle M, \in\rangle$ is isomorphism. If $W = L_\kappa[E]$, then by 2 there is transitive $W'$ and $E' \subseteq W'$ with $\langle W, \in, E\rangle \prec \langle W', \in, E'\rangle$. If $\gamma = \mathrm{ORD}^{W'}$, then $W' = L_\gamma[E']$ by elementarity. Using the club $C$ of $\alpha < \kappa$ with $\langle L_\alpha[E \cap \alpha], \in, E \cap \alpha\rangle \prec \langle L_\kappa[E], \in, E\rangle$ as additional predicate, if necessary, we may assume $\mathrm{cof}(\gamma) > \omega$, and thus that $E'$ is well-founded on $\gamma$. If $\pi^* : \langle \gamma, E'\rangle \to \langle N, \in\rangle$ is the Mostowski collapse, then the map $\pi^* \circ \pi^{-1}$ is an elementary embedding from $M$ to $N$ with critical point $\kappa$. □

We now discuss the relation to a characterization of weakly compact cardinals, as certain compactness properties for infinitary languages:

$L_{\kappa,\lambda}$ is the infinitary language with conjunctions and disjunctions of length $<\kappa$ and quantification over sequences of length $<\lambda$.

## Definition

We say that $\kappa$ has the *weak* $(\kappa, \lambda)$*-compactness property* if every $<\kappa$-satisfiable theory of size $\kappa$ in an $L_{\kappa,\lambda}$-language is satisfiable. We denote this property with $\mathrm{CP}(\kappa, \lambda)$.

## Theorem (folklore; embedding iff $\kappa$-compactness, for inaccessible $\kappa$)

*Assume that $\kappa$ is inaccessible. Then the following are equivalent.*

1. $\mathrm{EP}(\kappa)$: *For every transitive set $M$ of size $\kappa$ with $\kappa \in M$, there is transitive set $N$ and elementary embedding $j : M \to N$ with critical point $\kappa$*
2. $\mathrm{CP}(\kappa, \kappa)$: *Every $<\kappa$-satisfiable theory of size $\kappa$ in an $L_{\kappa,\kappa}$-language is satisfiable.*

## Proof.

1→2: Fix any $<\kappa$-satisfiable theory $T$ of size $\kappa$. Without loss, assume $T \subseteq \kappa$. Put $T$ into such $M$ with $M \prec H_{\kappa^+}$. Since $\kappa^{<\kappa} = \kappa$, it follows from an Skolem-Löwenheim argument that $H_{\kappa^+}$ sees that $T$ is $<\kappa$-satisfiable, and so $M$ sees this too. By 1, there is $j : M \to N$ with critical point $\kappa$ and $N$ transitive. Assume also that $N^{<\kappa} \subseteq N$. The theory $j(T)$ is $<j(\kappa)$-satisfiable in $N$, and since $T \in N$ has size $\kappa$ there, it follows that $T$ is satisfiable in $N$. And $N$ is correct about this, as $N^{<\kappa} \subseteq N$ ensures it has all witnessing sequences of length $<\kappa$. □

**Problem:** If, say, $2^\omega > \kappa$, then $N$ may not be closed under $\omega$-sequences, and so if $N$ thinks that $T$ is satisfiable, this won't mean that $T$ really is satisfiable. Such $N$ will only be correct about theories in $L_{\kappa,\omega}$. Thus: $\mathrm{EP}(\kappa)$ implies $\mathrm{CP}(\kappa, \omega)$.

## Theorem (folklore; embedding iff $\kappa$-compactness, for inaccessible $\kappa$)

*Assume that $\kappa$ is inaccessible. Then the following are equivalent.*

1. $\mathrm{EP}(\kappa)$: *For every transitive set $M$ of size $\kappa$ with $\kappa \in M$, there is transitive set $N$ and elementary embedding $j : M \to N$ with critical point $\kappa$*
2. $\mathrm{CP}(\kappa, \kappa)$: *Every $<\kappa$-satisfiable theory of size $\kappa$ in an $L_{\kappa,\kappa}$-language is satisfiable.*

## Proof.

2$\to$1: It is well known how to verify the extension property of a weakly compact cardinal, using the weak compactness property $\mathrm{CP}(\kappa, \kappa)$: Given any $A \subseteq V_\kappa$, let $T$ be the following theory:

- elementary diagram of $\langle V_\kappa, \in, A\rangle$
- the assertion that $\hat{\in}$ is well-founded. (as an $L_{\kappa,\omega_1}$-sentence)
- $c \neq \hat{a}$ for every $a \in V_\kappa$, where $c$ is a new constant.
- for every $a \in V_\kappa$, include the $L_{\kappa,\omega}$ sentence $\forall x (x \hat{\in} \hat{a} \iff \bigvee_{b \in a} x = \hat{b})$

This $L_{\kappa,\kappa}$ theory $T$ is $<\kappa$-satisfiable in $\langle V_\kappa, \in, A\rangle$, and has size $\kappa$. By $\mathrm{CP}(\kappa, \kappa)$, it is satisfiable, and the transitive collapse of any model of $T$ provides the desired proper extension $\langle V_\kappa, \in, A\rangle \prec \langle W', \in, A'\rangle$ with $W'$ transitive. □

Analyzing the proofs of the previous result give immediately the following:

### Theorem

*Let $\kappa$ be an uncountable cardinal. Then:*

1. $\mathrm{EP}(\kappa)$ *implies* $\mathrm{CP}(\kappa, \omega)$.
2. $\mathrm{CP}(\kappa, \omega_1)$ *implies* $\mathrm{EP}(\kappa)$.

### Proof.

1: If $T$ is a $L_{\kappa,\omega}$ theory of size $\kappa$, and $j : M \to N$ such that $N$ thinks that $T$ is satisfiable, then it really is.

2: We verify the extension property $\mathrm{XP}(\kappa)$. Thus, let $W \subseteq H_\kappa$ be transitive of size $\kappa$ and fix any $A \subseteq W$. As before, let $T$ be the $L_{\kappa,\omega_1}$ theory that includes the elementary diagram of $\langle W, \in, A\rangle$, the assertion that $\hat{\in}$ is well-founded, a sentence asserting that a new constant symbol will be interpreted as an element different from all $a \in W$, and for each element $a \in W$ an $L_{\kappa,\omega}$ sentence detailing all the elements of $a$. Again, $T$ is $<\kappa$-satisfiable and has size $\kappa$, and is thus satisfiable by $\mathrm{CP}(\kappa, \omega_1)$; the transitive collapse of any model of $T$ provides a proper extension $\langle W, \in, A\rangle \prec \langle W', \in, A'\rangle$ with $W'$ transitive, as desired. □

It is natural to ask:

Question: Is $\mathrm{EP}(\kappa)$ is strong enough to imply $\mathrm{CP}(\kappa, \omega_1)$?
Answer: No!

Question: Is $\mathrm{CP}(\kappa, \omega)$ strong enough to imply $\mathrm{EP}(\kappa)$?
Answer: Yes!

These answers are closely related to results in William Boos' dissertation ['71] under Kenneth Kunen. Boos writes that his argument was suggested by the proof of Keisler's extension property characterization of weakly compact cardinals, and indeed, his proof closely mimics Keisler's proof of the following result:

## Theorem (H. J. Keisler, '71)

*An inaccessible cardinal $\kappa$ is weakly compact if and only if for every $A \subseteq V_\kappa$, the model $\langle V_\kappa, \in, A\rangle$ has a proper elementary end-extension.*

- If $\mathcal{A} = \langle A, E\rangle$ and $\mathcal{B} = \langle B, F\rangle$ are (not necessarily well-founded) models of $\mathrm{ZFC}$, then $\mathcal{B}$ is an end-extension of $\mathcal{A}$ iff whenever $a \in A$, $b \in B$ and $bFa$, then $b \in A$.
- Note that Keisler's characterization does not insist on a well-founded extension, but only on a proper elementary end-extension.
- Being an end-extension of $\mathcal{A} = \langle A, E\rangle$ can be ensured using $L_{\kappa,\omega}$ sentences detailing all the elements of all elements of $A$! No need for $L_{\kappa,\omega_1}$ to ensure well-foundedness.

## Definition

A cardinal $\kappa$ has the *ordinal extension property*, if whenever $A \subseteq \kappa$, then there is an ordinal $\gamma > \kappa$ and a relation $A' \subseteq \gamma$ such that $\langle \kappa, \in, A \rangle \prec \langle \gamma, \in, A' \rangle$.

A cardinal $\kappa$ is *ordinal $\Pi^1_1$ indescribable*, if whenever $A \subseteq \kappa$ and $\varphi$ is a $\Pi^1_1$ sentence such that $\langle \kappa, \in, A \rangle \models \varphi$, then there is $\delta < \kappa$ such that $\langle \delta, \in, A \cap \delta \rangle \models \varphi$.

These properties were considered e.g. by A. Lévy ('71), and by W. Boos ('76).

## Theorem (W. Boos, '76)

*The following are equivalent for any $\kappa > \omega$:*

1. $\mathrm{CP}(\kappa, \omega)$
2. $\kappa$ *has the ordinal extension property.*
3. $\kappa$ *is ordinal $\Pi^1_1$ indescribable.*

## Theorem (W. Boos, '76)

*It is relatively consistent for any regular $\lambda < \kappa$ that $\mathrm{CP}(\kappa, \lambda)$ holds, but $\mathrm{CP}(\kappa, \lambda^+)$ fails.*

The following lemma connects these properties to our embedding property $\mathrm{EP}(\kappa)$:

## Lemma

*If $\kappa > \omega$ and $\kappa$ has the ordinal extension property, then $\kappa$ has the extension property $\mathrm{XP}(\kappa)$.*

In summary, we obtain the following:

## Theorem

*The following are equivalent for any $\kappa > \omega$:*

1. *$\kappa$ has the embedding property $\mathrm{EP}(\kappa)$*
2. *$\kappa$ has the extension property $\mathrm{XP}(\kappa)$*
3. *the weak compactness property $\mathrm{CP}(\kappa, \omega)$ holds*
4. *$\kappa$ has the ordinal extension property.*
5. *$\kappa$ is ordinal $\Pi^1_1$ indescribable.*

**THANK YOU!**