Generalized descriptive set theory with very large cardinals

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- We can generalize this result by considering sets of reals in the structure $L(\mathbb{R})$.
- $L(\mathbb{R})$ is the structure created by building L 'on top of the reals $\mathbb{R}$'. So $L_0 = \mathbb{R}$, $L_{\alpha+1}(\mathbb{R}) = \mathrm{Def}(L_\alpha(\mathbb{R}))$ and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$

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- The above generalizes to: assuming enough large cardinals, every set of reals in $L(\mathbb{R})$ has the perfect set property (Woodin).

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- Our main goal is to generalize the above situation to the structure $L(V_{\lambda+1})$. That is, we want to show that similar regularity properties hold in $L(V_{\lambda+1})$, and we want to find a 'fundamental regularity property' for $L(V_{\lambda+1})$.

large cardinals and elementary embeddings

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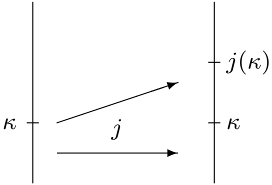
- We let $\kappa = \text{crit}\,(j)$ the critical point of the embedding j, which is the least κ such that $j(\kappa) \neq \kappa$. In fact $j(\kappa) > \kappa$.



measurable and strong cardinals

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- 2-strong cardinals are much stronger than measurable cardinals. For instance if κ is 2-strong then κ is a limit of measurable cardinals.
- In general, the more M agrees with V, the stronger the large cardinal.

the strongest large cardinals

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(AC) There is no (non-trivial) elementary embedding

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Definition

1. I_1 is the statement: for some λ, there exists an elementary embedding

$$j : V_{\lambda+1} \to V_{\lambda+1}.$$

2. I_3 is the statement: for some λ, there exists an elementary embedding

$$j : V_\lambda \to V_\lambda.$$

the axiom I_0

Definition (Woodin)

I_0 is the statement: there exists a λ such that there is an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with $\mathrm{crit}\,(j) < \lambda$.

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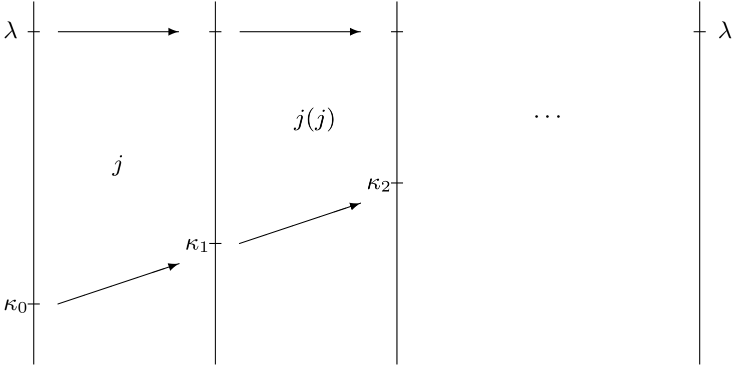
Woodin originally introduced I_0 in order to show that AD holds in $L(\mathbb{R})$ assuming large cardinals.

rank into rank embeddings

If $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary then λ is the sup of the *critical sequence* of j. That is, for $\kappa_0 = \operatorname{crit}(j)$ and $\kappa_{i+1} = j(\kappa_i)$ for $i < \omega$, we have

$$\lambda = \sup_{i<\omega} \kappa_i.$$



relationship with $L(\mathbb{R})$

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- So $L(\mathbb{R}) = L(V_{\omega+1})$ and $L(V_{\lambda+1})$ are both structures of the form $L(V_{\alpha+1})$ for α a strong limit of cofinality ω.

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- Furthermore, if AD holds in $L(\mathbb{R})$, then it does not satisfy the axiom of choice. And if I_0 holds at λ then $L(V_{\lambda+1})$ does not satisfy the axiom of choice.

relationship with $L(\mathbb{R})$

- If $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary and crit $(j) < \lambda$ then λ is the sup of the critical sequence of j. So $\mathrm{cof}(\lambda) = \omega$.
- So $L(\mathbb{R}) = L(V_{\omega+1})$ and $L(V_{\lambda+1})$ are both structures of the form $L(V_{\alpha+1})$ for α a strong limit of cofinality ω.
- Furthermore, if AD holds in $L(\mathbb{R})$, then it does not satisfy the axiom of choice. And if I_0 holds at λ then $L(V_{\lambda+1})$ does not satisfy the axiom of choice.
- Do $L(\mathbb{R})$ and $L(V_{\lambda+1})$ have similar structural properties? For instance does an analogue of the perfect set property hold in $L(V_{\lambda+1})$?

relationship with $L(\mathbb{R})$

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Let $\Theta = \Theta_\lambda = \sup\{\alpha |\, (\text{there exists a surjection of } V_{\lambda+1} \text{ onto } \alpha)^{L(V_{\lambda+1})}\}$.

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Theorem

Assume AD holds in $L(\mathbb{R})$. Then $L(\mathbb{R})$ satisfies the following:

1. *ω_1 is measurable. In fact the club filter is an ultrafilter on ω_1 (Solovay).*
2. *Θ is a limit of measurable cardinals (Moschovakis).*

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Theorem (Woodin)

Assume I_0 holds at λ. Then the following hold in $L(V_{\lambda+1})$.

1. *λ^+ is measurable.*
2. *Θ is a limit of measurable cardinals.*

perfect set property

Theorem (Davis)

Assume $L(\mathbb{R})$ satisfies AD. Then every set of reals in $L(\mathbb{R})$ has the perfect set property. That is if $X \subseteq \mathbb{R}$ and $X \in L(\mathbb{R})$ then either X is countable or X contains a perfect set and hence $|X| = 2^\omega$.

Theorem (C.)

Assume I_0 holds at λ. Then every subset $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ has the λ-splitting perfect set property. That is either $|X| \leq \lambda$ or X contains a λ-splitting perfect set and hence $|X| = 2^\lambda$.

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Shi and Woodin originally showed the perfect set property for sets in $L_\lambda(V_{\lambda+1})$ using very different techniques, which we will discuss later.

the club filter on λ^+

1. Woodin showed that in $L(V_{\lambda+1})$ the club filter restricted to some stationary set is an ultrafilter on λ^+. In fact, he showed that there is a partition $\langle T_\alpha | \alpha < \beta \rangle$ of λ^+ into stationary sets such that $\beta < \lambda$ and for all $\alpha < \beta$, the club filter restricted to T_α is an ultrafilter.

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Theorem (C.)

Assume I_0 at λ. Then there are no disjoint stationary subsets T_1, T_2 of S_ω (in V) such that $T_1, T_2 \in L(V_{\lambda+1})$.

partition relations on λ^+

Theorem (Woodin)

Suppose I_0 holds at λ. Then for all $\alpha < \beta < \omega_1$,

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2. Since ω_1-DC holds in $L(V_{\lambda+1})$, we have that in $L(V_{\lambda+1})$

$$\lambda^+ \not\to (\lambda^+)^{\omega_1}.$$

So it is not clear how to define a 'strong partition property' for $L(V_{\lambda+1})$.

analog of AD for $L(V_{\lambda+1})$

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Definition

For $X \subseteq V_{\lambda+1}$, let $I_0(X)$ be the statement that there exists an elementary embedding

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$$

with $\text{crit}\,(j) < \lambda$.

- We have

$$AD \to \text{the perfect set property}$$

but

$$I_0(X) \nrightarrow \text{the } \lambda\text{-splitting perfect set property.}$$

inverse limit reflection

We will introduce a property called 'inverse limit reflection' (ILR) such that if I_0 holds at λ then $L(V_{\lambda+1})$ satisfies ILR. Furthermore

$$\text{ILR} \rightarrow \text{the } \lambda\text{-splitting perfect set property.}$$

So ILR is in this sense a better analog of AD for $L(V_{\lambda+1})$ than I_0.

reflecting I_3 and I_1

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- Does some large cardinal axiom reflect I_3, I_1, and I_0? Yes.

Theorem

1. *(I_1 reflects I_3) Suppose there is $V_{\lambda+1} \to V_{\lambda+1}$ an elementary embedding. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}} \to V_{\bar{\lambda}}$ (Martin).*
2. *(I_0 reflects I_1) Suppose there is $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ an elementary embedding with crit$(j) < \lambda$. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}+1} \to V_{\bar{\lambda}+1}$ (Woodin).*
3. *Assume there exists $j : L_{\lambda^+ + \omega+1}(V_{\lambda+1}) \to L_{\lambda^+ + \omega+1}(V_{\lambda+1})$ elementary. Then there exists a $\bar{\lambda} < \lambda$ such that there is an elementary embedding $k : L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1}) \to L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1})$ with crit$(k) < \bar{\lambda}$ (Laver).*

Laver used a technique called ‘inverse limits’ to get his reflection result.

reflecting I_0

Theorem (C.)

($I_0^\#$ reflects I_0) Assume there exists an elementary embedding

$$j : L(V^\#_{\lambda+1}) \to L(V^\#_{\lambda+1})$$

with crit$(j) < \lambda$. Then there exists a $\bar{\lambda} < \lambda$ and an elementary embedding

$$k : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$$

with crit$(k) < \bar{\lambda}$.

The proof uses inverse limits as well.

definition of inverse limits

Definition (Laver)

An inverse limit $(J, \langle j_i | \, i < \omega\rangle)$ is a tuple such that the following hold:

1. For all $i < \omega$, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.
2. $\operatorname{crit}(j_0) < \operatorname{crit}(j_1) < \operatorname{crit}(j_2) < \cdots < \lambda$.
3. $\sup_{i<\omega} \operatorname{crit}(j_i) = \bar{\lambda} < \lambda$.
4. $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ is defined by: for all $a \in V_{\bar{\lambda}}$,

$$J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a) = (j_0 \circ j_1 \circ \cdots)(a).$$

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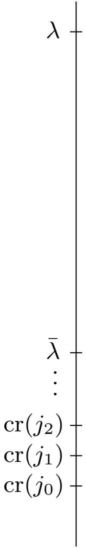
- If $(J, \langle j_i | \, i < \omega \rangle)$ is an inverse limit then we write

$$J = j_0 \circ j_1 \circ \cdots .$$

- We can rewrite an inverse limit as a direct limit as follows:

$$J = \cdots \circ j_0(j_1(j_2)) \circ j_0(j_1) \circ j_0.$$

picture of an inverse limit



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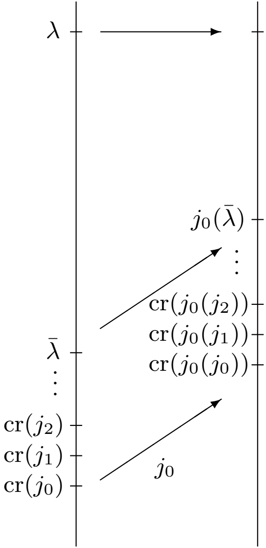

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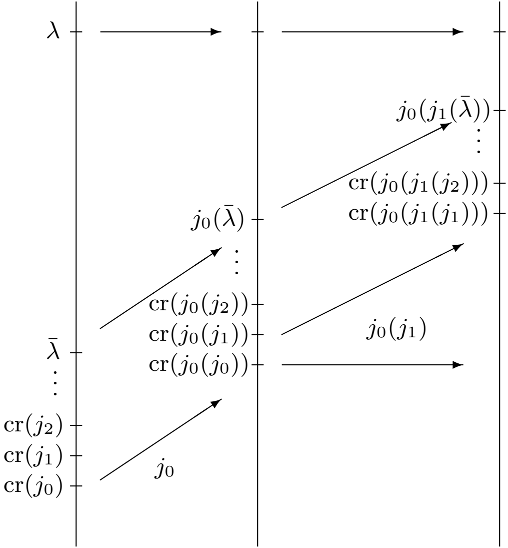

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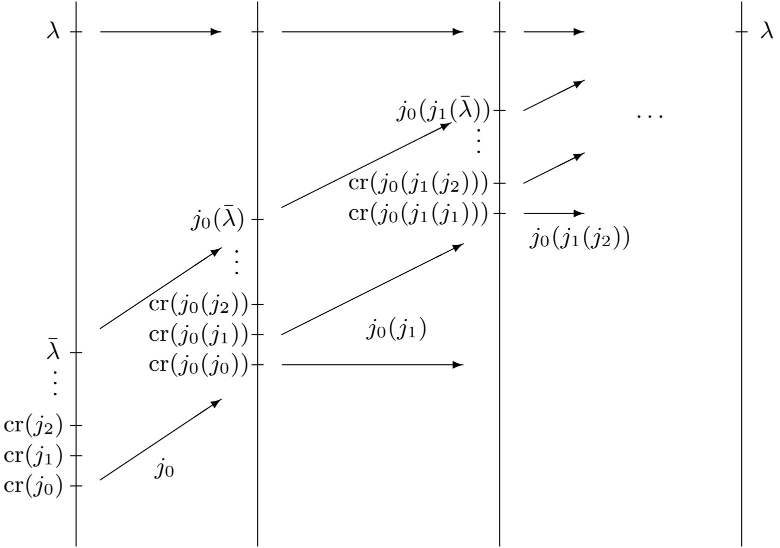

properties of inverse limits

- There are many theorems on inverse limits which take the basic form:

$$\text{property X for the embeddings } k_i \text{ for all } i < \omega \\ \Rightarrow \text{property } X \text{ for } K = k_0 \circ k_1 \circ \cdots$$

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  We say that property X *transfers* to inverse limits.
- For instance for (certain) inverse limits $K = k_0 \circ k_1 \circ \cdots$ we have for any $a \in V_{\lambda+1}$

$$\forall i < \omega (a \in \operatorname{rng} k_i) \rightarrow a \in \operatorname{rng} K.$$

inverse limit roots

- For $j, k : V_{\lambda+1} \to V_{\lambda+1}$ elementary embeddings k is a *square root* of j if $k(k \restriction V_\lambda) = j \restriction V_\lambda$.

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- If j is a strong embedding then it has lots of square roots: for instance if $j : L_1(V_{\lambda+1}) \to L_1(V_{\lambda+1})$ is elementary, then for any $a, b \in V_{\lambda+1}$, there is a square root $k : V_{\lambda+1} \to V_{\lambda+1}$ of j such that

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- If $K^0, K^1, \dots$ is such that K^{i+1} is an inverse limit root of K^i for all $i < \omega$, then this sequence converges to an inverse limit K^ω.
- For E a set of inverse limits, $\mathrm{CL}(E)$ is the set of inverse limits $J = j_0 \circ j_1 \circ \cdots$ such that for all $n < \omega$ there is $K = k_0 \circ k_1 \circ \cdots \in E$ with $(k_0, \dots, k_n) = (j_0, \dots, j_n)$.

inverse limit reflection

Definition

Inverse limit reflection at α is the statement that there is a collection E of inverse limits satisfying the following.

1. E is closed under taking inverse limit roots in the sense that for all $J \in E$ and $x \in V_{\lambda+1}$, there is $K \in E$ an inverse limit root of J such that $x \in \operatorname{rng} K$.
2. The property 'extension to $L_\alpha(V_{\lambda+1})$' transfers to inverse limits on $\mathrm{CL}(E)$. In fact, there are unique $\bar{\alpha}$ and $\bar{\lambda}$ such that for all $J \in \mathrm{CL}(E)$, J extends to an elementary embedding

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Inverse limit reflection is the statement that inverse limit reflection holds for all $\alpha < \Theta$.

inverse limit reflection

Theorem

Suppose I_0 holds at λ.

1. *Inverse limit reflection holds at λ^+ (Laver).*
2. *$L(V_{\lambda+1})$ satisfies inverse limit reflection(C.).*

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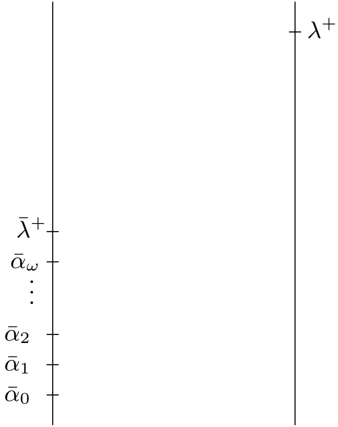
Inverse limit reflection seems to be one of the key tools for studying $L(V_{\lambda+1})$, and it was the key tool for the proof of the perfect set property and the result on non-splitting stationary subsets of λ^+. How to extend this property to hierarchies above $L(V_{\lambda+1})$ is still an open problem.

proof of no disjoint stationary sets

Theorem (C.)

Assume I_0 at λ. Then there are no disjoint stationary subsets T_1, T_2 of S_ω (in V) such that $T_1, T_2 \in L(V_{\lambda+1})$.



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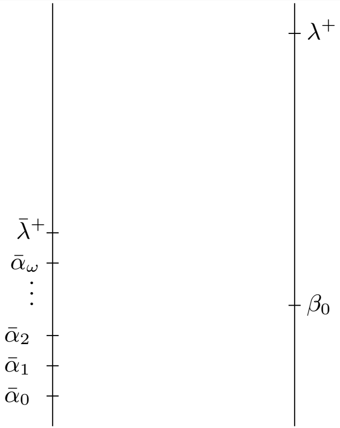

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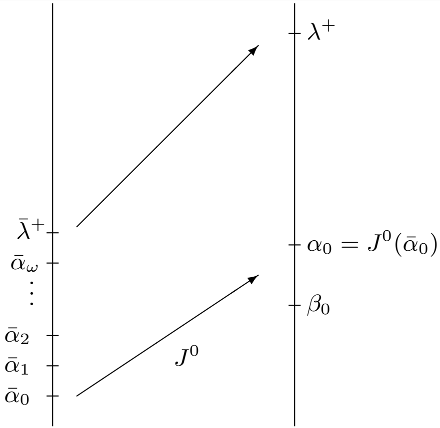

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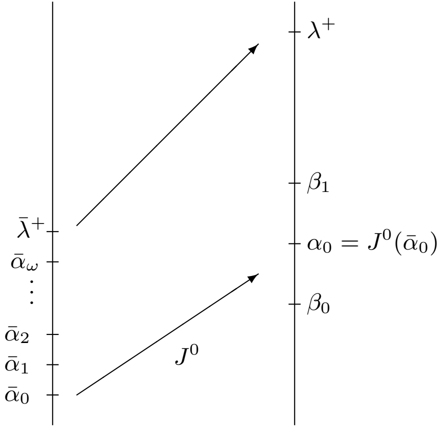

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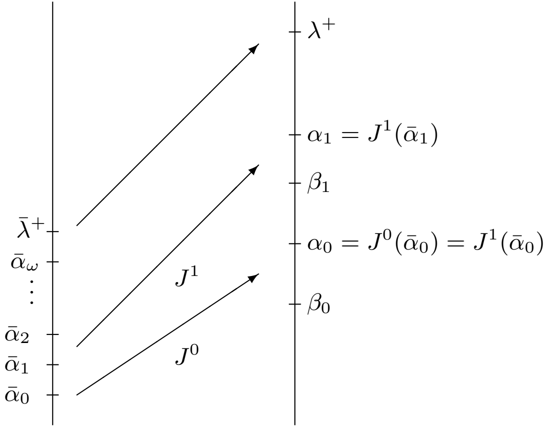

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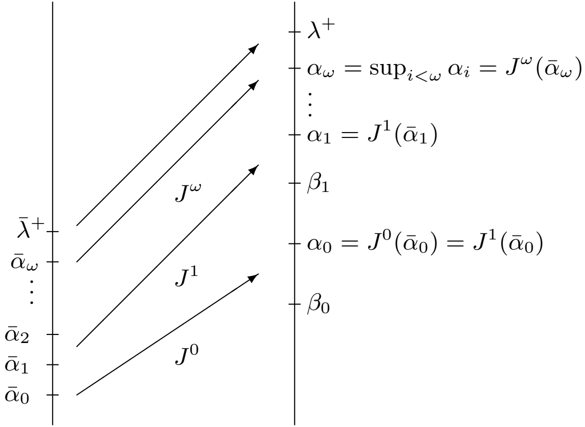

$U(j)$-representations

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- In fact, Woodin showed that $U(j)$-representations give even stronger properties for $L(V_{\lambda+1})$ than those which have been proven using inverse limits, such as a certain generic absoluteness result. However, it still remains unclear how many subsets of $V_{\lambda+1}$ have $U(j)$-representations.

$U(j)$-representations

Theorem (Woodin)

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary with crit$(j) < \lambda$. Then every set $X \subseteq V_{\lambda+1}$, $X \in L_\lambda(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$.

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The proof uses inverse limit techniques along with theorems of Woodin on $U(j)$-representations.

how far do $U(j)$-representations go?

Because the collection of $U(j)$-representations is closed under complements, along with the above theorems, we would expect the following.

Conjecture ($U(j)$-conjecture)

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then every set $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$.

This would be a stark contrast to scales in $L(\mathbb{R})$, but note that $U(j)$-representations seem not to be related to uniformization.

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Conjecture

Inverse limit reflection $\Rightarrow$ $U(j)$-representations exist (locally).

consequences of the $U(j)$-conjecture

Theorem (Dimonte-Friedman, Woodin independently)

If the $U(j)$-conjecture holds then if I_0 is consistent, it is consistent to have I_0 at some λ and the Singular Cardinal Hypothesis fails at λ.

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LSA is a very strong determinacy axiom.

Theorem (Woodin)

If the $U(j)$-conjecture holds then under enough large cardinals, after collapsing λ to ω there is a natural model of determinacy which satisfies LSA. In fact the Θ of this model is equal to $\Theta^{L(V_{\lambda+1})}$.

The above theorems already show that models of AD do not 'peter out' at the level of I_0.