Joint Laver diamonds

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Laver diamonds

Laver diamonds are prediction principles, connected with large cardinals, that significantly strengthen the usual $\diamondsuit_\kappa$ principle in terms of the sets which can be guessed and how well/often they are guessed.

Proto-definition

If κ is a large cardinal with a suitable elementary embedding definition, a function $\ell\colon \kappa \to V$ is a *Laver diamond* (or $\LaTeX$-style $\triangle_\kappa$ function) if for any reasonable target set x there is a large cardinal embedding j such that $j(\ell)(\kappa) = x$.

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Even if Laver diamonds do not exist outright, we can often force their existence.

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Two Laver diamonds

Suppose κ is measurable and ℓ is a Laver diamond for κ. Let ℓ_0, ℓ_1 be defined by $\ell_i(\xi) = \ell(\xi)(i)$.
Given any two targets $x_0, x_1 \in H_{\kappa^+}$, we have an embedding j such that $j(\ell)(\kappa) = (x_0, x_1)$. Therefore $j(\ell_i)(\kappa) = x_i$.

Joint Laver diamonds

Proto-definition (H., Hamkins)

Suppose κ is a large cardinal with a notion of Laver diamonds. A *joint Laver diamond* of length λ (or $\mathbb{\Delta}_{\kappa,\lambda}$ sequence) is a sequence of Laver diamonds $\vec{\ell} = \langle \ell_\alpha; \alpha < \lambda \rangle$ such that for any sequence of targets $\vec{x}$ there is an embedding j for which $j(\ell_\alpha)(\kappa) = x_\alpha$ for all α.

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- If $\kappa \leq \theta < 2^\kappa$ and there is a Laver diamond, are there joint Laver diamonds of lengths $> \theta$?
- If $\kappa \leq \theta < 2^\kappa$, does $◺_{\kappa,2^\kappa}$ have nontrivial consistency strength?

Consistency strength of long joint Laver diamonds

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Proof sketch.

We force with $\mathbb{P} = \mathbb{P}_\kappa * \mathrm{Add}(\kappa, 2^\kappa)$, where $\mathbb{P}_\kappa$ is a suitable preparatory iteration. If g is the final Cohen generic, then the slices of g code Laver diamonds.

- If $2^\kappa \leq \theta$, it is easy to build a master condition and run the usual lifting argument.
- If $\theta < 2^\kappa$, master conditions do not exist but we can build a 'master filter' using a technique of Magidor.

□

Separating short and long joint Laver diamonds

Fact

If κ is measurable and has a joint Laver diamond of length λ then there are at least 2^λ many normal measures on κ.

So we might try controlling the lengths of joint Laver diamonds by controlling the number of measures.

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Theorem (Apter-Cummings-Hamkins)

If κ is measurable then after forcing with $\mathbb{P} = \mathrm{Add}(\omega, 1) * \mathrm{Coll}\big(\kappa^+, 2^{2^\kappa}\big)$ there are at most κ^+ many normal measures on κ.

In particular, κ is still measurable in the extension but cannot have a joint Laver diamond of length κ^+.

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First pass to a model where κ has a Laver diamond and then force with $\mathrm{Add}(\omega, 1) * \mathrm{Coll}\big(\kappa^+, 2^{2^\kappa}\big)$.

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Theorem

If κ is measurable and $\kappa < \lambda \leq 2^\kappa$ is regular then there is a forcing extension in which $◬_{\kappa,\lambda}$ fails but $◬_{\kappa,<\lambda}$ holds.

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Theorem

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This argument will collapse 2^κ to λ. However, different methods, using work of Friedman and Magidor, allow for the nonexistence of joint Laver sequences of some length $\lambda < 2^\kappa$.

Separation for θ-supercompacts

Apter, Cummings and Hamkins also give analogous methods of controlling the number of measures on $\mathcal{P}_\kappa\theta$.

Theorem

If κ is θ-supercompact, θ is regular and $\theta^{<\kappa} = \theta$ then there is a forcing extension in which κ has a Laver diamond but no joint Laver diamonds of length θ^+.

Similar separation can be obtained at any point between θ and 2^κ.

θ-strongness joint Laver diamonds

Let us look briefly at θ-strong cardinals and their Laver diamonds.
For most θ there is not much difference between a single Laver diamond and a joint one.

Proposition

Let κ be θ-strong with $\kappa \cdot \omega \leq \theta$ and let $\lambda \leq 2^\kappa$ be a cardinal. If κ has a Laver diamond and θ is either a successor or $\lambda < \mathrm{cf}(\theta)$ then κ has a joint Laver diamond of length λ.

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Furthermore, we can force the existence of a Laver diamond under the same assumptions on θ.

Pathology

However, for more pathologic θ, the existence of even the shortest θ-strongness joint Laver diamonds exceeds θ-strongness in consistency strength.

Proposition

Let κ be θ-strong, where $\mathrm{cf}(\theta) = \omega$, and suppose that κ has a joint Laver diamond of length ω. Then there are stationarily many $\lambda < \kappa$ which are $(\lambda + \omega)$-strong.

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Proof.

A (κ, V_θ)-extender can be coded as a subset of V_θ and broken up into ω many pieces. Guessing these pieces yields a θ-strongness embedding $j\colon V \to M$ where κ is θ-strong in M (so, in particular, $(\kappa + \omega)$-strong). This reflects below κ in V. □

The problem with measures

Every known method for controlling the existence of Laver diamonds and joint Laver sequences for measurables relies on controlling the number of witnessing embeddings, i.e. measures. It seems inconceivable that this is the only obstruction, but the question remains open.

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Question

Suppose κ is measurable and has many normal measures (say 2^{2^κ}). Are there joint Laver diamonds for κ? Or is it possible that there simply are no Laver diamonds at all?

Thank you.

Thank you.

Happy birthday Joel!