Minimal models of second-order set theories

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A classical result

Theorem (Shepherdson, Cohen)

There is a least transitive model of ZFC.

Gödel–Bernays set theory with global choice

Models of GBC consist of sets $(x, y, z, \ldots)$ and classes $(X, Y, Z, \ldots)$. For a model $(M, \mathcal{X})$ we will always assume that $\mathcal{X} \subseteq \mathcal{P}(M)$.

The axioms:

- ZFC for sets;
- Extensionality for classes: classes with the same members are equal.
- Class Replacement: if F is a class function and a is a set then $F''a$ is a set.
- Comprehension for formulae with only set quantifiers: $\{x : \varphi(x, P)\}$ is a class.
- Global Choice: there is a bijection $V \to \mathrm{Ord}$.

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Theorem (Folklore)

Every countable model of ZFC *can be extended to a model of* GBC *without adding any new sets.* *(Force to add a global well-order, close off under definability.)*

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Definition

$(M, \mathcal{X})$ is transitive if $\mathcal{X}$ is transitive or, equivalently, if M is transitive.

Stronger second-order set theories

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Theorem (Folklore)

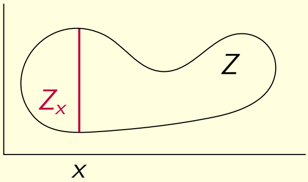
KM *proves there is a truth predicate for first-order truth. In particular,* KM *proves* Con(ZFC).

Stronger second-order set theories

KM^+ is KM plus the Class Collection axiom schema:

- If for every set x there is a class Y so that $\varphi(x, Y, P)$, then there is a class Z so that $\varphi(x, Z_x, P)$ for every x, where Z_x is the slice

$$Z_x = \{y : (x, y) \in Z\}.$$



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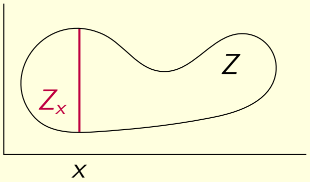

Fact

If κ is inaccessible then $(V_\kappa, V_{\kappa+1})$ is a model of KM^+.

Second-order set theory is first-order set theory in disguise

Theorem

KM^+ *and* $\mathsf{ZFC}^-_{\mathsf{I}}$ *are bi-interpretable, where* $\mathsf{ZFC}^-_{\mathsf{I}}$ *is* $\mathsf{ZFC} - \mathsf{Powerset}$ *plus the assertion that there is a largest cardinal, which is inaccessible.*

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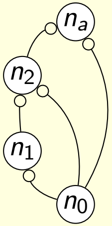
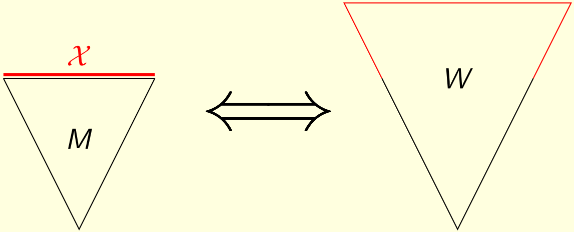


represents the set $a = \{0, 2\}$.

Unrolling the second-order model



$(M, \mathcal{X}) \models \mathsf{KM}^+$ $\quad$ $W \models \mathsf{ZFC}^-_\mathsf{I}$

L in the second-order part

We can build L inside the second-order part of a model of KM, keeping the same sets.

Theorem

The classes of any model of KM *can be shrunk to produce a model of* KM^+ *with the same sets.*

The main theorem

Theorem (W.)

There is no least transitive model of KM.

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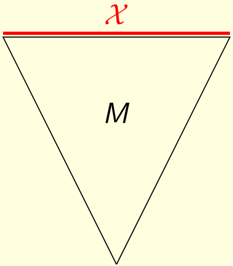

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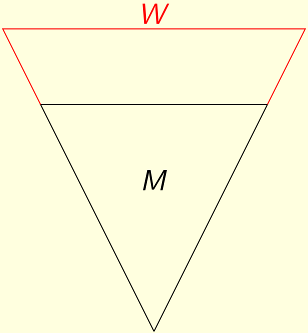


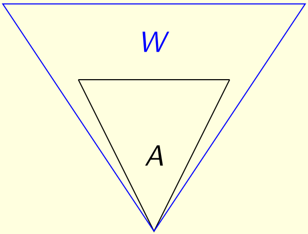
A detour through a theorem of Harvey Friedman

Theorem (H. Friedman)

- *A is countable, admissible ($=$ transitive model of KP).*
- *T is an L_A theory which is Σ_1-definable in A.*
- *T has an admissible model $W \supseteq A$.*

Then, there is $U \models T + \mathrm{KP}$ so that

- $\mathrm{wfp}(U) \supseteq A$ *and*
- $\mathrm{Ord}^{\mathrm{wfp}(U)} = \mathrm{Ord}^A$.



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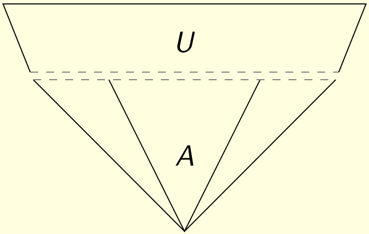

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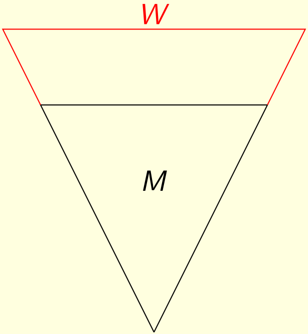

W is transitive.

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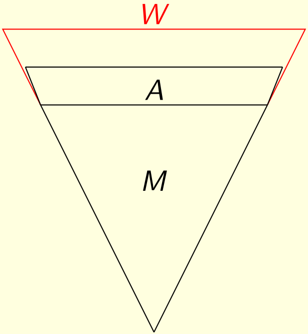

Find $A \in W$ admissible with $M \in A$.

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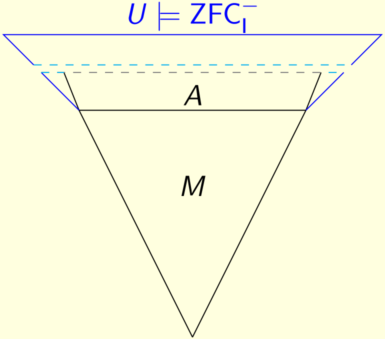

Apply Friedman's theorem to A.

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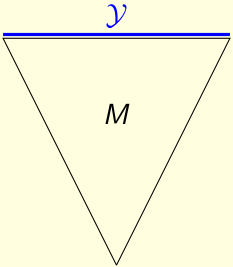
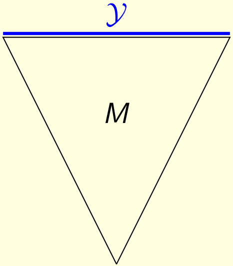


Turn the $\mathsf{ZFC}_\mathsf{I}^-$ model into a KM^+ model $(M, \mathcal{Y})$.

Suppose that our $(M, \mathcal{X})$ is correct about well-foundedness.



$\mathcal{X} \not\subseteq \mathcal{Y}$ because $\mathcal{Y}$ doesn't have any element representing Ord^A.

What if $(M, \mathcal{X})$ is wrong about well-foundedness?

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Some corollaries

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$M \models \mathrm{ZFC}$. Say $\mathcal{X} \subseteq \mathcal{P}(M)$ is a KM-*realization for M* if $(M, \mathcal{X})$ is a model of KM.

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M a countable model of ZFC*. There is no least* KM*-realization for M.*

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Theorem

M a countable model of ZFC*. There is a least* GBC*-realization for M if and only if M has a definable global well-order*

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Theorem (W.)

M a countable model of ZFC. *There is no least* KM-*realization for M.*

Theorem

M a countable model of ZFC. *There is a least* GBC-*realization for M if and only if M has a definable global well-order if and only if M is a model of $\exists x\ V = \mathrm{HOD}(x)$.*

Recent work by others

Theorem (C. Antos & S. Friedman)

For any real r there is a least β-model of $\mathsf{KM}^+ + \mathsf{DC}_\infty$ containing r.

Definition

$(M, \mathcal{X})$ is a β*-model* if it is correct about well-foundedness.

Some open questions

- What about the uncountable case? Can an uncountable model of ZFC have a least KM-realization? A least GBC-realization?

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Fact

There are uncountable models of ZFC *without any* GBC*-realizations.*

Consider $M \models \mathsf{ZFC} + \forall x\ V \neq \mathrm{HOD}(x)$ *which is rather classless.*

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- What if we ask for *minimal*, rather than *least* models?

Thank you!

Some references:

- Carolin Antos & Sy-David Friedman. Hyperclass forcing in Morse-Kelley class theory. submitted
- Harvey Friedman. Countable models of set theories. In A.R.D. Mathias & H. Rogers, editors, *Cambridge Summer School in Mathematical Logic*, pages 539–573. New York, Springer-Verlag, 1973.
- Kameryn J Williams. Minimal models of second-order set theories. in preparation