Bounding, splitting and almost disjointness can be quite different

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The cardinal invariants of the continuum arise from various combinatorial, topological and algebraic properties of the real line. They are usually defined as the minimum size of a set of reals, satisfying certain property, and take cardinal values μ, where $\aleph_0 < \mu \leq \mathfrak{c}$.

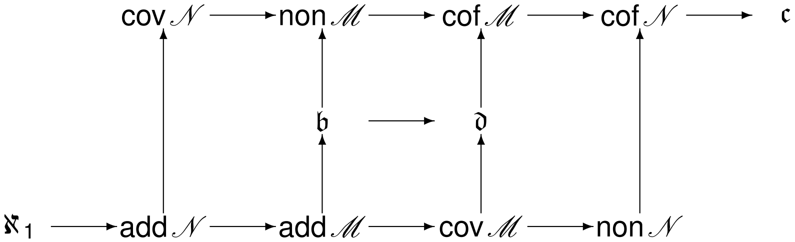
- A family $\mathscr{A} \subseteq [\omega]^\omega$ is almost disjoint, if its members have pairwise finite intersections. An infinite a.d. family, is a mad family, if it is maximal (with respect to this property) under inclusion. The almost disjointness number $\mathfrak{a}$, is defined as the minimal size of a maximal almost disjoint family.
- A family $\mathscr{B} \subseteq {}^\omega\omega$ is unbounded, if for every $f \in {}^\omega\omega$ there is $g \in \mathscr{B}$ such that $g \not\leq^* f$. The bounding number is defined as the minimal size of an unbounded family.
- A family $\mathscr{D} \subseteq {}^\omega\omega$ is dominating, if for every $f \in {}^\omega\omega$ there is $g \in \mathscr{D}$ such that $f \leq^* g$. The dominating number, $\mathfrak{d}$, is defined as the minimal size of a dominating family.

Let $\mathscr{I}$ be an ideal of subsets of X. Then define:

- $\operatorname{add}(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \text{ and } \bigcup \mathscr{A} \notin \mathscr{I}\}$,
- $\operatorname{cov}(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \text{ and } \bigcup \mathscr{A} = X\}$,
- $\operatorname{non}(\mathscr{I}) = \min\{|Y| : Y \subseteq Y \text{ and } Y \notin \mathscr{I}\}$, and
- $\operatorname{cof}(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \text{ and } \forall B \in \mathscr{I} \exists A \in \mathscr{A} (B \subseteq A)\}$.

We are interested in the ideals $\mathscr{N}$ of all measure zero sets and $\mathscr{M}$ of all meager sets on the real line.

Cichoń's diagram



- $\mathfrak{b} \leq \mathfrak{d}$, $\mathfrak{b} \leq \mathfrak{a}$
- $\mathfrak{d}$ and $\mathfrak{a}$ are independent

Theorem

Assume CH and let $\lambda = \lambda^\omega$ be a cardinal. Then

$$V^{\mathbb{C}_\lambda} \vDash (\mathfrak{a} = \aleph_1 \textit{ and } \mathfrak{d} = \mathfrak{c} = \lambda).$$

Proof:

Anticipating all Cohen names for reals, recursively construct in V a mad family, which remains maximal in $V^{\mathbb{C}}$. Using the fact that every $\mathbb{C}_\lambda$-name for a real, is a $\mathbb{C}$-name for a real and that $\mathbb{C}$ adds an unbounded real, we obtain the desired result.

Theorem (Folklore)

Assume CH and let $\lambda = \lambda^\omega$ be a cardinal. Then in $V^{\mathbb{C}_\lambda}$ every mad family has size either $\aleph_1$ or $\mathfrak{c} = \lambda$.

Proof:

Let $\omega_2 \cdot 2 \leq \kappa < \lambda$ and $\{\dot{A}_\alpha\}_{\alpha<\kappa}$ names for a.d. sets in $[\omega]^{\leq\omega}$. For $n \in \omega$ let $\{p^\alpha_{n,i}\}_{i\in\omega}$ be a max antichain deciding "$n \in \dot{A}_\alpha$". That is, there are $\{k^\alpha_{n,i}\}_{n,i\in\omega} \subseteq \{0,1\}$ such that

$$p^\alpha_{n,i} \Vdash n \in \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 1 \text{ and } p^\alpha_{n,i} \Vdash n \notin \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 0.$$

Let $B^\alpha = \bigcup_{n,i\in\omega} \mathrm{dom}(p^\alpha_{n,i})$. Then $|\bigcup_\alpha B^\alpha| \leq \kappa < \lambda$.

By CH, Δ-system Lemma, wlg the $\{B_\alpha\}_{\alpha<\omega_2}$ form a Δ-system with root R. Let $\phi_{\alpha,\beta} : B^\alpha \to B^\beta$ be a bijection fixing R. Then $\phi_{\alpha,\beta}$ induces an $\psi_{\alpha,\beta} : \mathbb{C}_{B^\alpha} \cong \mathbb{C}_{B^\beta}$, and so an isomorphism between $\mathbb{C}_{B^\alpha}$-names and $\mathbb{C}_{B^\beta}$-names. There are only $2^{\aleph_0} = \aleph_1$ isomorphism types of names and so we may assume that $\psi_{\alpha,\beta}(p^\alpha_{n,i}) = p^\beta_{n,i}$ and $k^\alpha_{n,i} = k^\beta_{n,i} (=: k_{n,i})$, e.g. $\psi_{\alpha,\beta}$ identifies $\dot{A}^\alpha$ with $\dot{A}^\beta$, and $k^\alpha_{n,i}$ does not depend on α.

Define $\dot{A}^\kappa$ as follows. Let $B^\kappa \cap \bigcup_{\alpha<\kappa} B^\alpha = R$ and for $\alpha < \omega_2$ let $\phi_{\alpha,\kappa} : B^\alpha \to B^\kappa$ be a bijection fixing R. Let $\psi_{\alpha,\kappa} : \mathbb{C}_{B^\alpha} \cong \mathbb{C}_{B^\kappa}$ be induced by $\phi_{\alpha,\kappa}$, $p^\kappa_{n,i} := \psi_{\alpha,\kappa}(p^\alpha_{n,i})$, $k^\kappa_{n,i} := k_{n,i}$.

Let $\beta < \kappa$. There is $\alpha < \omega_2$ such that $B^\beta \cap B^\alpha \subseteq R$, and so $B^\kappa \cap B^\beta = B^\alpha \cap B^\beta$. Thus the canonical mapping extending $\phi_{\alpha,\kappa}$ and sending $B^\alpha \cup B^\beta$ to $B^\kappa \cup B^\beta$ is a bijection, inducing an isomorphism $\mathbb{C}_{B^\alpha \cup B^\beta} \cong \mathbb{C}_{B^\kappa \cup B^\beta}$. Therefore $\Vdash_{\mathbb{C}_\lambda}$ "$\dot{A}^\kappa \cap \dot{A}^\beta$ is finite" and so $\{\dot{A}_\alpha\}_{\alpha<\kappa}$ is not maximal. $\square$

How to obtain the consistency of $\mathfrak{d} < \mathfrak{a}$?

- The technique of finite support iteration of ccc posets, can not be used, since at stages of countable cofinality, one adds Cohen and so unbounded reals.
- Another approach would be to use countable support iteration of proper ${}^\omega\omega$-bounding posets, since such iterations are known to preserve the ground model reals as a dominating family, and so as a witness to $\mathfrak{d} = \omega_1$. The question then reduces to the following. Given a mad family $\mathscr{A}$, is there a proper ${}^\omega\omega$-bounding poset which adds a real almost disjoint from $\mathscr{A}$?
- How about the consistency of $\mathfrak{d} < \mathfrak{a}$, where $\aleph_2 < \mathfrak{a} \leq \mathfrak{c}$?

Let κ be measurable, $\mathscr{U}$ a κ complete ultrafilter and $\mathbb{P}$ a ccc poset. Then $\mathbb{P}^\kappa/\mathscr{U}$ destroys the maximality of any a.d. family of size $\geq \kappa$ from the $\mathbb{P}$-extension (note $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathscr{U}$), while it preserves all scales of length different from κ. Therefore if $\lambda > \mu > \kappa$ regular, with $\lambda^\omega = \lambda$ and $\mathbb{P}$ is the finite support iteration of Hechler forcing of length μ, then taking ultrapowers λ times, with special care in the limit steps, will produce the consistency of $\mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$.

However the existence of a measurable is not necessary for obtaining the above consistency. In fact the existence of such a measurable can be substituted, in a certain sense, by an isomorphism of names argument.

Template

Let $(L,\leq)$ be a l.o. and for $x \in L$ let $L_x := \{y \in L : y < x\}$. A template is a pair $\mathscr{T} = ((L,\leq),\mathscr{I})$ where $\mathscr{I} \subseteq \mathscr{P}(L)$ is closed under finite intersections and unions, contains $\emptyset, L$,

- if $x, y \in L$, $y \in L$ and $x < y$ then $\exists A \in \mathscr{I}(A \subseteq L_y \wedge x \in A)$,
- if $A \in \mathscr{I}$, $x \in L \backslash A$, then $A \cap L_x \in \mathscr{I}$,

and $\mathscr{I}$ is well-founded (under $\subseteq$) with a rank function $\mathrm{Dp} : \mathscr{I} \to \mathbb{ON}$. Denote $\mathrm{Rk}(\mathscr{T}) = \mathrm{Dp}(L)$.

Indexed template

An indexed template is a pair $\langle L, \bar{\mathscr{I}} \rangle$, where $\bar{\mathscr{I}} := \langle \mathscr{I}_x \rangle_{x \in L}$ such that L is a linear order, $\mathscr{I}_x \subseteq \mathscr{P}(L_x)$ for all $x \in L$ and

1. $\varnothing \in \mathscr{I}_x$,
2. $\mathscr{I}_x$ is closed under finite unions and intersections,
3. $\mathscr{I}_x \subseteq \mathscr{I}_y$ if $x < y$ and
4. $\mathscr{I}(L) := \bigcup_{x \in L} \mathscr{I}_x \cup \{L\}$ is well-founded by the subset relation and witnessed by a rank function denoted Dp.

If $\langle L, \mathscr{I} \rangle$ is a template, then $\langle L, \bar{\mathscr{I}} \rangle$ is an indexed template, where $\bar{\mathscr{I}} = \langle \mathscr{I}_x \rangle_{x \in L}$ for $\mathscr{I}_x := \{A \in \mathscr{I} : A \subseteq L_x\}$.

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Let $(L, \bar{\mathscr{I}})$ be an indexed template.

- For $x \in L$ let $\hat{\mathscr{I}}_x$ be the ideal (on L_x) generated by $\mathscr{I}_x$;
- For $A \subseteq X$, $\mathscr{I}_x \restriction A = \{A \cap B : B \in \mathscr{I}_x\}$ is the trace of $\mathscr{I}_x$ on A;
- Let $\bar{\mathscr{I}} \restriction A := \langle \mathscr{I}_x \restriction A \rangle_{x \in A}$, $\mathscr{I}(A) = \bigcup_{x \in A} \mathscr{I}_x \restriction A \cup \{A\}$.

Then for each $A \subseteq L$, $\langle A, \bar{\mathscr{I}} \restriction A \rangle$ is an indexed template and $\mathscr{I}(A)$ is well-founded.

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Given an indexed template $(L, \bar{\mathscr{I}})$ for $X \subseteq L$ define $\mathrm{Dp}(X) := \mathrm{rank}_{\mathscr{I}(X)}(X)$. Then $\mathrm{Dp} : \mathscr{P}(L) \to$ **ON** is monotone and

1. if $\exists x \in Y$ s.t. $X \in \mathscr{I}_x \restriction Y$ then $\mathrm{Dp}(X) < \mathrm{Dp}(Y)$;
2. if in addition $X \subsetneq Y \cap L_x$, then also $\mathrm{Dp}(X \cup \{x\}) < \mathrm{Dp}(Y)$.

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Iteration of Hechler posets along an indexed template

Let $\langle L, \bar{\mathscr{I}} \rangle$ template. For $A \subseteq L$, recursively on $\mathrm{Dp}(A)$, define a poset $\mathbb{P}{\restriction}A$ as follows:

1. $p \in \mathbb{P}{\restriction}A$ iff p is a finite sequence of names such that $\operatorname{dom} p \subseteq A$ and either $p = \varnothing$ or, if $x = \max(\operatorname{dom} p)$, then there exists a $B \in \mathscr{I}_x{\restriction}A$ such that $p{\restriction}L_x \in \mathbb{P}{\restriction}B$ and $p(x)$ is a $\mathbb{P}{\restriction}B$-name for a condition in $\mathbb{D}^{V^{\mathbb{P}{\restriction}B}}$.

The extension relation

Let $p, q \in \mathbb{P}\restriction A$, let $x := \max(\operatorname{dom} p)$, $y := \max(\operatorname{dom} q)$ in case p, q are non-empty. Then $q \leq_A p$ if $\operatorname{dom} p \subseteq \operatorname{dom} q$ and either $p = \emptyset$, or

1. $x = y$ and $\exists B \in \mathscr{I}_y\restriction A$ such that $p\restriction L_y, q\restriction L_y \in \mathbb{P}\restriction B$, $q\restriction L_y \leq_B p\restriction L_y$ and $p(y), q(y)$ are $\mathbb{P}\restriction B$-names for conditions in $\mathbb{D}^{V^{\mathbb{P}\restriction B}}$ such that $q\restriction L_y$ forces $q(y) \leq p(y)$, or
2. $x < y$ and $\exists B \in \mathscr{I}_y\restriction A$ such that $p, q\restriction L_y \in \mathbb{P}\restriction B$, $q\restriction L_y \leq_B p$ and $q(y)$ is a $\mathbb{P}\restriction B$-name for a condition in $\mathbb{D}^{V^{\mathbb{P}\restriction B}}$.

In either case B is called a witness to $q \leq p$.

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In either case B is called a witness to $q \leq p$.

Let $(L, \bar{\mathscr{I}})$ be an indexed template and $\mathbb{P}\restriction L$ an iteration of Hechler forcing along it.

1. Let $A \subset B$, $B \subseteq L$. Then $\mathbb{P}\restriction A \lessdot \mathbb{P}\restriction B$.
2. The post $\mathbb{P}\restriction L$ is ccc (and furthermore Knaster).
3. Let $x \in L$, $A \in \mathscr{I}$, $A \subseteq L_x$. Then $(\mathbb{P}\restriction A) * \dot{\mathbb{D}} \lessdot \mathbb{P}\restriction L$.
4. For any $p \in \mathbb{P}\restriction L$ there is a countable $A \subseteq L$ such that $p \in \mathbb{P}\restriction A$. If $\dot{f}$ is a $\mathbb{P}\restriction L$-name for a real then there is a countable $A \subseteq L$ such that $\dot{f}$ is a $\mathbb{P}\restriction A$-name.

Examples:

- Let $L = \mu (= L_\mu)$ be an ordinal, $\mathscr{I} = \{L_\alpha = \alpha : \alpha \leq \mu\} = \mu + 1$. Then $\mathbb{P}{\restriction}L$ is the μ-stage finite support iteration of $\mathbb{D}$.
- Let L be arbitrary and let $\mathscr{I} = [L]^{<\omega} \cup \{L\}$. Then $(L, \mathscr{I})$ is a template and $\mathbb{P}{\restriction}L$ adds a family of functions in ${}^\omega\omega$ which is canonically order isomorphic to L.

Theorem (Shelah)

Assume CH. Let $\lambda > \mu > \aleph_1$ and $\lambda = \lambda^\omega$ be regular cardinals. Then there is a template $L(\mu, \lambda)$ such that in the generic extension obtained by iterating Hechler forcing along it,

$$\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda.$$

$L = L(\mu,\lambda)$

Let λ^* be a disjoint copy of λ with the reverse ordering. Then $L = L(\mu,\lambda)$ consists of all non-empty finite sequences x such that $x(0) \in \mu$ and $x(n) \in \lambda^* \cup \lambda$ for $n > 0$.

Define $x < y$ iff either $x(0) < y(0)$, or

- $x \subset y$ and $y(|x|) \in \lambda$, or
- $y \subset x$ and $x(|y|) \in \lambda^*$, or
- for $n := \min\{m : x(m) \neq y(m)\} > 0$, either $x(n) \in \lambda^*$ and $y(n) \in \lambda$, or both are in λ and $x(n) <_\lambda y(n)$, or both are in λ^* and $x(n) >_{\lambda^*} y(n)$.

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$\mathscr{I} = \mathscr{I}(\mu,\lambda)$

To define the template structure on L, choose a seq. $\Sigma = \langle S_\alpha \rangle_{\alpha<\omega_1}$ of pairwise disjoint, coinitial subsets of λ^* such that $\lambda^* = \bigcup \Sigma$.

- Say $x \in L$ to be relevant if $|x| \geq 3$ is odd, $x(n) \in \lambda^*$ for odd n, and $x(n) \in \lambda$ for even n, $x(|x|-1) \in \omega_1$, and whenever $n < m$ are even such that $x(n), x(m) < \omega_1$, then there are $\beta < \alpha$ such that $x(n-1) \in S^\alpha$, $x(m-1) \in S^\beta$.
- For relevant x set $J_x = [x{\restriction}[|x|-1), x)$. If $x < y$ are relevant, then either $J_y \cap J_x = \emptyset$, or $J_x \subset J_y$.

Then $\mathscr{I}(\mu,\lambda)$ is consists of all finite unions of sets from $\{L_\alpha : \alpha \leq \mu\} \cup \{J_x : x \text{ relevant}\} \cup \{\{x\} : x \in L\}$.

Short scales

Let G be $\mathbb{P}$-generic and let f_α be the Hechler generic added in coordinate $\alpha < \mu$.

- Let f be a real in $V[G]$. Then $\exists A \in [L]^{\leq\omega}$ s.t. $f \in V[G \cap \mathbb{P}\restriction A]$. Since $\mu \subseteq L$ is regular, uncountable $\exists \alpha < \mu (A \subseteq L_\alpha)$. Thus $f \in V[\mathbb{P}\restriction L_\alpha \cap G]$. Since $L_\alpha \in \mathscr{I}$, $\mathbb{P}\restriction L_\alpha * \dot{\mathbb{D}} \lessdot \mathbb{P}$, and so $f \leq^* f_\alpha$. Therefore $\langle f_\alpha : \alpha < \mu \rangle$ is dominating, and so $\mathfrak{d} \leq \mu$.
- If $F \subseteq \omega^\omega$ is a family of size $< \mu$ in $V[G]$, then there must be some $\alpha < \mu$ such that all reals of F already are in $V[G \cap \mathbb{P}\restriction L_\alpha]$, and so f_α dominates F. Thus $\mu \leq \mathfrak{b}$.

Therefore $\mu \leq \mathfrak{b} \leq \mathfrak{d} \leq \mu$.

Isomorphism of names

Let $\kappa \geq \omega \cdot 2$, $\mu \leq \kappa < \lambda$ and let $\{\dot{A}^\alpha\}_{\alpha<\kappa}$ be names for a.d. sets in $[\omega]^{\leq\omega}$. For $n \in \omega$ let $\{p^\alpha_{n,i}\}_{i\in\omega}$ be a max antichain deciding "$n \in \dot{A}_\alpha$". That is, there are $\{k^\alpha_{n,i}\}_{i,n\in\omega} \subseteq \{0,1\}$ such that

$$p^\alpha_{n,i} \Vdash n \in \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 1 \text{ and } p^\alpha_{n,i} \Vdash n \notin \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 0.$$

Then $B^\alpha = \bigcup_{n,i\in\omega} \mathrm{dom}(p^\alpha_{n,i})$ is at most countable.

By CH, Δ-system Lemma, wlg $\{B^\alpha\}_{\alpha<\omega_2}$ form a Δ-system with root R. Let $\phi_{B^\alpha,B^\beta} : B^\alpha \to B^\beta$ be a bijection fixing R. It induces $\psi_{B^\alpha,B^\beta} : \mathbb{P}{\restriction}B^\alpha \cong \mathbb{P}{\restriction}B^\beta$, which extends to an isomorphism between $\mathbb{P}{\restriction}B^\alpha$ and $\mathbb{P}{\restriction}B^\beta$-names. There are only $2^{\aleph_0} = \aleph_1$ isomorphism types of names, we may suppose that ψ_{B^α,B^β} identifies $\dot{A}^\alpha$ with $\dot{A}^\beta$, e.g. $\psi(p^\alpha_{n,i}) = p^\beta_{n,i}$ and $k^\alpha_{n,i} = k^\beta_{n,i} := k_{n,i}$

The properties of $L_{\mu,\lambda}$ allow to obtain an appropriate $B^\kappa \subseteq L$ such that for all $\alpha < \omega_1$, $\mathbb{P}\restriction B^\kappa \cong \mathbb{P}\restriction B^\alpha$. Define $\dot{A}^\kappa$ to be the image of $\dot{A}^\alpha$ under this isomorphism. Let $\beta < \kappa$. Then there is $\alpha < \omega_1$ such that $\mathbb{P}\restriction B^\kappa \cup B^\beta \cong \mathbb{P}\restriction B^\alpha \cup B^\beta$ and the same isomorphism sends $\dot{A}^\kappa$ to $\dot{A}^\alpha$. Since $\dot{A}^\alpha$ and $\dot{A}^\beta$ are forced to be almost disjoint, so are $\dot{A}^\kappa$ and $\dot{A}^\beta$. Thus $\mathscr{A}$ is not maximal. □

Preservation of splitting

- Baumgartner and Dordal proved that the finite support iteration of Hechler forcing over a model of CH preserves a witness to $\mathfrak{s} = \aleph_1$.
- Their preservation theorem has an analogue in the context of template iterations. Thus if we add ω_1-Cohen reals $\mathscr{C}$ to a model of CH and iterate $\mathbb{D}$ along $L(\mu, \lambda)$, the set $\mathscr{C}$ remains a splitting family in the final generic extension and so a witness to $\mathfrak{s} = \aleph_1$ in $V^{\mathbb{P}\restriction L}$.

Assume CH.

- (J. Brendle, 2003) It is relatively consistent that $\text{cof}(\mathfrak{a}) = \omega$.
- (V.F., A.Törnquist, 2013) It is relatively consistent that $\text{cof}(\mathfrak{a}_g) = \omega$.

Furthermore, we obtain a uniform proof to the consistency of each of $\mathfrak{a}$, $\mathfrak{a}_g$, $\mathfrak{a}_p$, $\mathfrak{a}_e$ being of countable cofinality by axiomatizing Brendle's technique of iteration along a two-sided template. Such constructions allow for a product-like forcing to be completely embedded into the template forcing iteration.

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Theorem (Shelah)

Assume CH. Let $\lambda > \mu > \aleph_1$, $\lambda = \lambda^\omega$ be regular.

1. *There is a forcing extension satisfying* $\text{add}(\mathcal{N}) = \text{cov}(\mathcal{N}) = \aleph_1$, $\mathfrak{b} = \mathfrak{d} = \mu$, $\text{non}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \lambda = \mathfrak{c}$.
2. *There is a forcing extension satisfying* $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \mu$ *and* $\mathfrak{a} = \mathfrak{c} = \lambda$.
3. *There is a forcing extension satisfying* $\mathfrak{a} = \aleph_1$, $\text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \mu$, $\mathfrak{a}_g = \mathfrak{c} = \lambda$.

Proof:

Let $\mathscr{T}$ be the template $(L(\mu,\lambda),\mathscr{I}(\mu,\lambda))$.

1. $\mathbb{P}(\mathscr{T},\mathbb{D})$.
2. $\mathbb{P}(\mathscr{T},\mathbb{L})$.
3. $\mathbb{P}(\mathscr{T},\mathbb{E})$.

$\square$

Problem

Is $\aleph_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ consistent?

Iteration along an indexed template template

Let θ be a regular uncountable cardinal, $\langle L, \bar{\mathscr{I}} \rangle$ indexed template, H, M disjoint sets with $L = H \cup M$. For each $x \in M$ let $C_x \in \hat{\mathscr{I}}_x$ be fixed of size $< \theta$.For $A \subseteq L$, recursively on $\mathrm{Dp}(A)$, define a poset $\mathbb{P}{\restriction}A$ and for each $x \in A$ and $B \in \hat{\mathscr{I}}_x{\restriction}A$, a $\mathbb{P}{\restriction}B$-name $\dot{\mathbb{Q}}_x^B$ as follows:

1. If $x \in H$ then $\dot{\mathbb{Q}}_x^B$ is a $\mathbb{P}{\restriction}B$-name for $\mathbb{D}^{V^{\mathbb{P}{\restriction}B}}$.
2. If $x \in M$, for a fixed $\mathbb{P}{\restriction}C_x$-name $\dot{F}_x$ for a filter base of size $< \theta$, $\dot{\mathbb{Q}}_x^B = \mathbb{M}_{\dot{F}_x}$ if $C_x \subseteq B$, or name for the trivial poset otherwise.
3. $p \in \mathbb{P}{\restriction}A$ iff p is a finite sequence of names such that $\mathrm{dom}\, p \subseteq A$ and either $p = \varnothing$ or, if $x = \max(\mathrm{dom}\, p)$, then $\exists B \in \mathscr{I}_x{\restriction}A$ such that $p{\restriction}L_x \in \mathbb{P}{\restriction}B$ and $p(x)$ is a $\mathbb{P}{\restriction}B$-name for a condition in $\dot{\mathbb{Q}}_x^B$.

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The extension relation

Let $p, q \in \mathbb{P}\restriction A$ and let $x := \max(\operatorname{dom} p)$, $y := \max(\operatorname{dom} q)$ in case p, q are non-empty. Then $q \leq_A p$ if $\operatorname{dom} p \subseteq \operatorname{dom} q$ and either $p = \varnothing$, or

1. $x = y$ and $\exists B \in \mathscr{I}_y\restriction A$ such that $p\restriction L_y, q\restriction L_y \in \mathbb{P}\restriction B$, $q\restriction L_y \leq_B p\restriction L_y$ and $p(y), q(y)$ are $\mathbb{P}\restriction B$-names for conditions in $\dot{\mathbb{Q}}^B_y$ such that $q\restriction L_y$ forces in $\mathbb{P}\restriction B$ that $q(y) \leq p(y)$, or
2. $x < y$ and $\exists B \in \mathscr{I}_y\restriction A$ such that $p, q\restriction L_y \in \mathbb{P}\restriction B$, $q\restriction L_y \leq_B p$ and $q(y)$ is a $\mathbb{P}\restriction B$-name for a condition in $\dot{\mathbb{Q}}^B_y$.

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Note that for each $A \subseteq L$:

1. If $X \subseteq A$ then $\mathbb{P}{\restriction}X \lessdot \mathbb{P}{\restriction}A$;
2. $\mathbb{P}{\restriction}A$ is ccc;
3. if $p \in \mathbb{P}{\restriction}A$ and $\dot{x}$ is a $\mathbb{P}{\restriction}A$-name for a real, then there is $C \subseteq A$ of size $< \theta$ such that $p \in \mathbb{P}{\restriction}C$ and $\dot{x}$ is a $\mathbb{P}{\restriction}C$-name.

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Width of Shelah's template

Fix uncountable regular cardinals $\theta < \mu < \lambda$.

- For $\delta \leq \lambda$ let L^δ be the linear order $L(\lambda\mu, \delta)$ and
- let $\mathscr{I}^\delta$ be the template structure $\mathscr{I}(\lambda\mu, \delta)$.
- Then $\langle L^\delta, \bar{\mathscr{I}}^\delta \rangle$ is an indexed template, where $\mathscr{I}^\delta_x$ is the family $\{A \in \mathscr{I}^\delta : A \subseteq L^\delta_x\}$.

Eliminating small mad families

Assume $\theta^{<\theta} = \theta$ and $\lambda^{<\lambda} = \lambda$. Let $\theta^+ < \delta < \lambda$, $\mathbb{P}^\delta$ be an iteration of $\mathbb{D}$ and Mathias forcings of size $< \theta$ along L^δ and $\dot{A}$ a $\mathbb{P}{\restriction}L^\delta$-name of an a.d. family of size $\kappa \in (\theta, \lambda)$. Then, there is a $\delta < \delta' < \lambda$ and an iteration $\mathbb{P}^{\delta'}$ along $L^{\delta'}$ such that

1. $\mathbb{P}^\delta{\restriction}X = \mathbb{P}^{\delta'}{\restriction}X$ for all $X \subseteq L^\delta$,
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Theorem (V.F. and D. Mejía)

There is an iteration $\mathbb{P}^\lambda$ along L^λ that forces $\mathfrak{s} = \theta < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$.

- To get an upper bound for $\mathfrak{s}$ adjoint a family $\mathscr{C}$ of Cohen reals of size θ. The preservation theorem for the iteration of $\mathbb{D}$ along a template generalize to the iteration of Mathias posets of size $< \theta$ and $\mathbb{D}$ along a template to guarantee that $\mathscr{C}$ remains splitting in the final extension. Guaranteeing that all pseudo-bases of size $< \theta$ have been diagonalized implies $\theta \leq \mathfrak{s}$ and so $V^{\mathbb{P}\restriction L^\lambda} \vDash \mathfrak{s} = \theta$.

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$\mathfrak{s} = \theta < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$.

- To get an upper bound for $\mathfrak{s}$ adjoint a family $\mathscr{C}$ of Cohen reals of size θ. The preservation theorem for the iteration of $\mathbb{D}$ along a template generalize to the iteration of Mathias posets of size $< \theta$ and $\mathbb{D}$ along a template to guarantee that $\mathscr{C}$ remains splitting in the final extension. Guaranteeing that all pseudo-bases of size $< \theta$ have been diagonalized implies $\theta \leq \mathfrak{s}$ and so $V^{\mathbb{P}\restriction L^\lambda} \models \mathfrak{s} = \theta$.

- Using an appropriate book-keeping device, and extending λ-many times a given template of width $< \lambda$ to adjoint a real almost disjoint from the elements of a given mad family of size $< \lambda$, we obtain a template L^λ such that for an appropriate iteration $\mathbb{P}{\restriction}L^\lambda$, in $V^{\mathbb{P}\restriction L^\lambda}$ there are no m.a.d. families of size $< \mathfrak{c} = \lambda$.
- To provide $V^{\mathbb{P}\restriction L^\lambda} \vDash \mathfrak{b} = \mathfrak{d} = \mu$ it is sufficient to guarantee that $H \cap \{\langle\alpha\rangle : \alpha \in \lambda\mu\}$ is cofinal in L.

Some open questions

- Can we modify the above construction so that in addition $\mathfrak{a}$ is of countable cofinality?
- Is it consistent that $\aleph_1 < \mathfrak{a} < \mathrm{non}(\mathscr{M}) < \mathfrak{a}_g$?

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Thank you!