Virtual large cardinals

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This is joint work with Ralf Schindler (University of Münster).

Virtual large cardinals

Suppose $\mathcal{A}$ is a very large cardinal property, e.g.,

- supercompact,
- extendible,
- n-huge*,
- rank-into-rank,

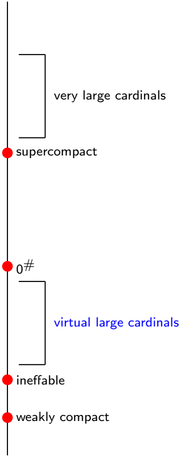
characterized by the existence of "suitable" set-sized embeddings.

(Certain closure requirements are not allowed.)

We say that a cardinal is virtually $\mathcal{A}$ if the embeddings of V-structures characterizing $\mathcal{A}$ exist in set-forcing extensions.

Virtual large cardinals are mini versions of their actual counterparts.

- Silver indiscernibles are virtual large cardinals.
- Virtual large cardinals are situated between ineffables and $0^{\#}$.
- Virtual large cardinals are downward absolute to L.



Generic versus virtual large cardinals

Generic large cardinals

Suppose $\mathcal{A}$ is a large cardinal property characterized by existence of elementary embeddings $j : V \to M$ (with additional properties).

A cardinal is generically $\mathcal{A}$ if the embeddings $j : V \to M$ characterizing $\mathcal{A}$ exist in set-forcing extensions (possibly with $M \not\subseteq V$).

- δ is generically supercompact if for every $\lambda > \delta$ in some set-forcing extension there is an elementary $j : V \to M$ with $\text{crit}(j) = \delta$ and $j\text{"}\lambda \in M$.
- A tiny cardinal such as ω_1 can be generically $\mathcal{A}$.
- Consistency strength of a generically $\mathcal{A}$ cardinal is usually on par with $\mathcal{A}$.

Virtual large cardinals

Suppose $\mathcal{A}$ is a large cardinal property characterized by existence of "suitable" set-sized embeddings.

A cardinal is virtually $\mathcal{A}$ if the embeddings of V-structures characterizing $\mathcal{A}$ exist in set-forcing extensions.

- Virtual large cardinals are actual large cardinals.
- Consistency strength of a virtually $\mathcal{A}$ cardinal is usually much weaker than $\mathcal{A}$.

Absoluteness Lemma for embeddings on countable structures

Suppose B and A are (first-order) structures in the same language.

Lemma: (Absoluteness Lemma for embeddings on countable structures)
Suppose B is countable and B elementarily embeds into A. If W is a (set or class) model of (a sufficiently large fragment of) ZFC such that

- $B, A \in W$,
- B is countable in W,

then B elementarily embeds into A in W.

Proof:

- Enumerate $B = \{b_n \mid n < \omega\}$ in W. Let $B \upharpoonright n = \{b_i \mid i < n\}$.
- Let T be the tree of all partial finite isomorphisms

$$f : B \upharpoonright n \to A$$

  ordered by extension.
- B elementarily embeds into A if and only if T has a cofinal branch.
- T is ill-founded in V, and hence in W. □

When do embeddings exist in a set-forcing extension? Part I

Suppose B and A are (first-order) structures in the same language.

Theorem: TFAE

1. B elementarily embeds into A in some set-forcing extension.
2. B elementarily embeds into A in $V^{\mathrm{Coll}(\omega,B)}$.

Proof:
(2) $\Rightarrow$ (1): Trivial.
(1) $\Rightarrow$ (2): Suppose a set-forcing extension $V[G]$ has an elementary $j : B \to A$.

- Let $|B|^V = \delta$.
- Consider a further extension $V[G][H]$ by $\mathrm{Coll}(\omega, \delta)$.
- $j \in V[G][H]$ and B is countable in $V[G][H]$.
- $V[H] \subseteq V[G][H]$ has some elementary $j^* : B \to A$ (by Absoluteness Lemma). $\square$

When do embeddings exist in a set-forcing extension? Part II

Suppose B and A are (first-order) structures in the same language.

Let $G(B, A)$ be an ω-length Ehrenfeucht-Fraïssé type game:

- Stage n: player I plays some $b_n \in B$ and player II plays some $a_n \in A$.
- Player II wins if for every $n \in \omega$ and formula $\varphi(x_0, \ldots, x_n)$,

$$B \models \varphi(b_0, \ldots, b_n) \leftrightarrow A \models \varphi(a_0, \ldots, a_n),$$

  and otherwise player I wins.
- If player II loses, she must do so in finitely many steps.
- $G(B, A)$ is closed, and hence determined by the Gale-Stewart Theorem.

Theorem: TFAE

1. Player II has a winning strategy in $G(B, A)$.
2. B elementarily embeds into A in $V^{\mathrm{Coll}(\omega, B)}$.

Proof:

(1) $\Rightarrow$ (2): A winning strategy for player II, remains winning in $V^{\mathrm{Coll}(\omega,B)}$ because no new finite sequences are added.

(2) $\Rightarrow$ (1): Fix $p \Vdash$ "$\tau : \check{B} \to \check{A}$ is an elementary embedding".

- To every finite $\vec{b}$ from B, associate $p_{\vec{b}} \Vdash \tau(\vec{b}) = \vec{a}$ below p so that: if $\vec{b}'$ extends $\vec{b}$, then $p_{\vec{b}'} \leq p_{\vec{b}}$.
- A winning strategy for player II: play $\vec{a}$ in response to $\vec{b}$. $\square$

Remarkable cardinals

Definition: (Schindler) A cardinal κ is remarkable if for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that in some set-forcing extension there is an elementary $j : V_{\bar{\lambda}}^V \to V_\lambda^V$ with $j(\text{crit}(j)) = \kappa$.

Theorem: (Schindler, '00) The assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing is equiconsistent with a remarkable cardinal.

Theorem: (Schindler, '15) The weak Proper Forcing Axiom, wPFA, is equiconsistent with a remarkable cardinal.

- wPFA implies $\text{PFA}_{\aleph_2}$ (PFA for antichains of size at most $\aleph_2$).
- wPFA is consistent with $\square_\delta$ for $\delta \geq \omega_2$, and hence wPFA does not imply $\text{PFA}_{\aleph_3}$.

Theorem: (Magidor, '71) A cardinal κ is supercompact if and only if for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$.

Remarkable cardinals are virtually supercompact!

Various virtual large cardinals

Definition:

- A cardinal κ is virtually extendible if for every $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha^V \to V_\beta^V$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.
- A cardinal κ is virtually n-huge* if for some $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha^V \to V_\beta^V$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.
- A cardinal κ is virtually rank-into-rank if for some $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha^V \to V_\alpha^V$ with $\text{crit}(j) = \kappa$.

Proposition: (Beyond Kunen's Inconsistency) A set-forcing extension can have an elementary $j : V_\alpha^V \to V_\alpha^V$ with $\alpha >> \lambda$, the supremum of the critical sequence.

Proof: If κ is a Silver indiscernible and $\alpha >> \kappa$ is uncountable in V, then there is an elementary $j : L_\alpha \to L_\alpha$ with $\text{crit}(j) = \kappa$.

Aside: n-huge*-cardinals

Problem: n-huge cardinals do not have a suitable embedding characterization for virtualization.

Definition: A cardinal κ is n-huge* if for some $\alpha > \kappa$, there is $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.

Proposition: An $n+1$-huge cardinal is n-huge* and an n-huge* cardinal is an n-huge limit of n-huge cardinals.

Proof:

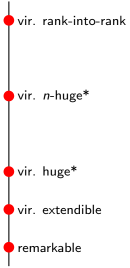
- If κ is $n+1$-huge, then there is $j : V_{j^n(\kappa)+1} \to V_{j^{n+1}(\kappa)+1}$.
- If κ is n-huge*, then there are cardinals $\kappa = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda$ and a κ-complete normal ultrafilter U over $P(\lambda)$ such that for each $i < n$,

$$\{x \in P(\lambda) \mid \text{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in U.$$

The hierarchy of virtual large cardinals

Proposition:

- A virtually extendible cardinal is a remarkable limit of remarkable cardinals.
- If κ is virtually huge*, then V_κ is a model of proper class many virtually extendible cardinals.
- If κ is virtually $n+1$-huge*, then V_κ is a model of proper class many virtually n-huge* cardinals.
- If κ is virtually rank-into-rank, then for every $n \in \omega$, V_κ is a model of proper class many virtually n-huge* cardinals.



α-iterable cardinals

Definition: A weak κ-model (for a cardinal κ) is a transitive $M \models \mathrm{ZFC}^-$ of size κ and height above κ.

Suppose M is a weak κ-model.

Proposition: TFAE.

- There exists an elementary $j : M \to N$ with $\mathrm{crit}(j) = \kappa$.
- There exists an M-ultrafilter U with a well-founded ultrapower.
  - U is an M-ultrafilter if $\langle M, \in, U\rangle \models U$ is a normal ultrafilter.
  - $U = \{A \in M \mid \kappa \in j(A)\}$.

Definition: An M-ultrafilter U is weakly amenable if for every $X \in M$ with $|X|^M \leq \kappa$, $X \cap U \in M$.

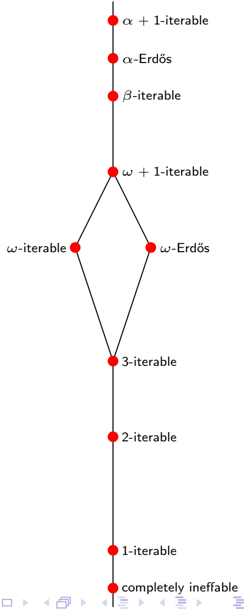
- U is partially internal to M.
- Weak amenability is needed to iterate the ultrapower construction.

Definition: (G.) A cardinal κ is α-iterable ($1 \leq \alpha \leq \omega_1$) if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable M-ultrafilter on κ with α-many well-founded iterated ultrapowers.

α-iterable cardinals in the hierarchy

Theorem: (G., Welch, '08)

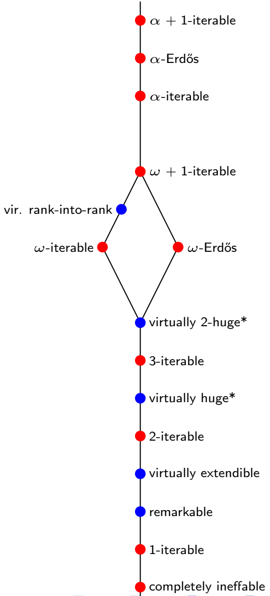
- A 1-iterable cardinal is a limit of completely ineffable cardinals.
- An α-iterable cardinal is a limit of β-iterable cardinals for every $\beta < \alpha$.
- Suppose α is additively indecomposable.
  - An α-Erdős cardinal implies the consistency of a proper class of β-iterable cardinals for every $\beta < \alpha$.
  - An $\alpha + 1$-iterable cardinal implies the consistency of an α-Erdős cardinal.



Virtual large cardinals in the hierarchy

Theorem: (G., Schindler, '15)

- A remarkable cardinal is a 1-iterable limit of 1-iterable cardinals.
- If κ is 2-iterable, then V_κ is a model of proper class many virtually extendible cardinals.
- A virtually n-huge* cardinal is an $n+1$-iterable limit of $n+1$-iterable cardinals.
- If κ is $n+2$-iterable, then V_κ is a model of proper class many virtually n-huge* cardinals.
- A virtually rank-into-rank cardinal is an ω-iterable limit of ω-iterable cardinals.
- An $\omega+1$-iterable cardinal implies the consistency of a virtually rank-into-rank cardinal.



Applications

Let $C^{(n)}$ be the class club of ordinals α such that $V_\alpha \prec_{\Sigma_n} V$.

Definition: (Bagaria) A cardinal κ is $C^{(n)}$-extendible if for every $\alpha > \kappa$, there is an extendibility embedding j with $j(k) \in C^{(n)}$.

Vopěnka's Principle: For every proper class $\mathcal{C}$ of structures of the same type, there are $B \neq A$, both in $\mathcal{C}$, such that B elementarily embeds into A.

Generic Vopěnka's Principle: (Bagaria, G., Schindler) For every proper class $\mathcal{C}$ of structures of the same type, there are $B \neq A$, both in $\mathcal{C}$, such that B elementarily embeds into A in some set-forcing extension.

Theorem: (Bagaria, G., Schindler, '15)

- Generic Vopěnka's Principle for $\mathbf{\Pi}_1$-definable classes is equiconsistent with a proper class of remarkable cardinals.
- Generic Vopěnka's Principle for $\mathbf{\Pi}_n$-definable classes ($n \geq 2$) is equiconsistent with a proper class of virtually $C^{(n)}$-extendible cardinals.

Questions

Question: Does a virtually rank-into-rank cardinal imply the consistency of an ω-Erdős cardinal?

Question: (vague) Can we define something akin to an algebra of embeddings for the virtually rank-into-rank cardinals?

Question: Applications?

Thank you!