Virtual large cardinals

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Virtual large cardinals

Suppose $\mathcal{A}$ is a very large cardinal property, e.g.,
- supercompact,
- extendible,
- $n$-huge$^*$,
- rank-into-rank,
characterized by the existence of "suitable" set-sized embeddings.
(Certain closure requirements are not allowed.)

We say that a cardinal is virtually $\mathcal{A}$ if the embeddings of $V$-structures characterizing $\mathcal{A}$ exist in set-forcing extensions.

Virtual large cardinals are mini versions of their actual counterparts.
- Silver indiscernibles are virtual large cardinals.
- Virtual large cardinals are situated between ineffables and $0^#$.  
- Virtual large cardinals are downward absolute to $L$. 

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Virtual large cardinals

Set Theory Day 3 / 16
Generic versus virtual large cardinals

Generic large cardinals

Suppose $\mathcal{A}$ is a large cardinal property characterized by existence of elementary embeddings $j : V \to M$ (with additional properties).

A cardinal is generically $\mathcal{A}$ if the embeddings $j : V \to M$ characterizing $\mathcal{A}$ exist in set-forcing extensions (possibly with $M \not\subseteq V$).

- $\delta$ is generically supercompact if for every $\lambda > \delta$ in some set-forcing extension there is an elementary $j : V \to M$ with $\text{crit}(j) = \delta$ and $j'' \lambda \in M$.
- A tiny cardinal such as $\omega_1$ can be generically $\mathcal{A}$.
- Consistency strength of a generically $\mathcal{A}$ cardinal is usually on par with $\mathcal{A}$.

Virtual large cardinals

Suppose $\mathcal{A}$ is a large cardinal property characterized by existence of “suitable” set-sized embeddings.

A cardinal is virtually $\mathcal{A}$ if the embeddings of $V$-structures characterizing $\mathcal{A}$ exist in set-forcing extensions.

- Virtual large cardinals are actual large cardinals.
- Consistency strength of a virtually $\mathcal{A}$ cardinal is usually much weaker than $\mathcal{A}$.
Absoluteness Lemma for embeddings on countable structures

Suppose $B$ and $A$ are (first-order) structures in the same language.

**Lemma**: (Absoluteness Lemma for embeddings on countable structures)
Suppose $B$ is countable and $B$ elementarily embeds into $A$. If $W$ is a (set or class) model of (a sufficiently large fragment of) ZFC such that

- $B, A \in W$,
- $B$ is countable in $W$,

then $B$ elementarily embeds into $A$ in $W$.

**Proof**:

- Enumerate $B = \{ b_n \mid n < \omega \}$ in $W$. Let $B \upharpoonright n = \{ b_i \mid i < n \}$.
- Let $T$ be the tree of all partial finite isomorphisms $f : B \upharpoonright n \rightarrow A$ ordered by extension.

$B$ elementarily embeds into $A$ if and only if $T$ has a cofinal branch.

$T$ is ill-founded in $V$, and hence in $W$. □
When do embeddings exist in a set-forcing extension? Part I

Suppose \( B \) and \( A \) are (first-order) structures in the same language.

**Theorem:** TFAE

1. \( B \) elementarily embeds into \( A \) in some set-forcing extension.
2. \( B \) elementarily embeds into \( A \) in \( V^{\text{Coll}(\omega, B)} \).

**Proof:**

(2) \( \Rightarrow \) (1): Trivial.

(1) \( \Rightarrow \) (2): Suppose a set-forcing extension \( V[G] \) has an elementary \( j : B \rightarrow A \).

- Let \( |B|^V = \delta \).
- Consider a further extension \( V[G][H] \) by \( \text{Coll}(\omega, \delta) \).
- \( j \in V[G][H] \) and \( B \) is countable in \( V[G][H] \).
- \( V[H] \subseteq V[G][H] \) has some elementary \( j^* : B \rightarrow A \) (by Absoluteness Lemma). \( \square \)
When do embeddings exist in a set-forcing extension? Part II

Suppose $B$ and $A$ are (first-order) structures in the same language.

Let $G(B, A)$ be an $\omega$-length Ehrenfeucht-Fraïssé type game:

- **Stage $n$:** player I plays some $b_n \in B$ and player II plays some $a_n \in A$.
- **Player II wins** if for every $n \in \omega$ and formula $\varphi(x_0, \ldots, x_n)$,
  $$B \models \varphi(b_0, \ldots, b_n) \leftrightarrow A \models \varphi(a_0, \ldots, a_n),$$
  and otherwise player I wins.
- If player II loses, she must do so in finitely many steps.
- $G(B, A)$ is closed, and hence determined by the Gale-Stewart Theorem.

**Theorem:** TFAE

1. Player II has a winning strategy in $G(B, A)$.
2. $B$ elementarily embeds into $A$ in $V^{\text{Coll}(\omega, B)}$.

**Proof:**

$(1) \Rightarrow (2)$: A winning strategy for player II, remains winning in $V^{\text{Coll}(\omega, B)}$ because no new finite sequences are added.

$(2) \Rightarrow (1)$: Fix $p \models "\tau : \bar{B} \rightarrow \bar{A} \text{ is an elementary embedding}"$.

- To every finite $\vec{b}$ from $B$, associate $p_{\vec{b}} \models \tau(\vec{b}) = \vec{a}$ below $p$ so that:
  - if $\vec{b}'$ extends $\vec{b}$, then $p_{\vec{b}'} \leq p_{\vec{b}}$.
- A winning strategy for player II: play $\vec{a}$ in response to $\vec{b}$. □

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Virtual large cardinals
Set Theory Day 7 / 16
Remarkable cardinals

**Definition**: (Schindler) A cardinal $\kappa$ is remarkable if for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that in some set-forcing extension there is an elementary $j : V_{\bar{\lambda}} \rightarrow V_{\lambda}$ with $j(\text{crit}(j)) = \kappa$.

**Theorem**: (Schindler, '00) The assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing is equiconsistent with a remarkable cardinal.

**Theorem**: (Schindler, '15) The weak Proper Forcing Axiom, wPFA, is equiconsistent with a remarkable cardinal.

- wPFA implies $\text{PFA}_{\aleph_2}$ (PFA for antichains of size at most $\aleph_2$).
- wPFA is consistent with $\square_\delta$ for $\delta \geq \omega_2$, and hence wPFA does not imply $\text{PFA}_{\aleph_3}$.

**Theorem**: (Magidor, '71) A cardinal $\kappa$ is supercompact if and only if for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that there is an elementary $j : V_{\bar{\lambda}} \rightarrow V_{\lambda}$ with $j(\text{crit}(j)) = \kappa$.

Remarkable cardinals are virtually supercompact!
Various virtual large cardinals

**Definition:**
- A cardinal $\kappa$ is virtually extendible if for every $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V^V_\alpha \rightarrow V^V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.
- A cardinal $\kappa$ is virtually $n$-huge* if for some $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V^V_\alpha \rightarrow V^V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.
- A cardinal $\kappa$ is virtually rank-into-rank if for some $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V^V_\alpha \rightarrow V^V_\beta$ with $\text{crit}(j) = \kappa$.

**Proposition:** (Beyond Kunen’s Inconsistency) A set-forcing extension can have an elementary $j : V^V_\alpha \rightarrow V^V_\alpha$ with $\alpha \gg \lambda$, the supremum of the critical sequence.

**Proof:** If $\kappa$ is a Silver indiscernible and $\alpha \gg \kappa$ is uncountable in $V$, then there is an elementary $j : L^V_\alpha \rightarrow L^V_\alpha$ with $\alpha \gg \lambda$, the supremum of the critical sequence.

**Aside:** $n$-huge*-cardinals

**Problem:** $n$-huge cardinals do not have a suitable embedding characterization for virtualization.

**Definition:** A cardinal $\kappa$ is $n$-huge* if for some $\alpha > \kappa$, there is $j : V^V_\alpha \rightarrow V^V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.

**Proposition:** An $n + 1$-huge cardinal is $n$-huge* and an $n$-huge* cardinal is an $n$-huge limit of $n$-huge cardinals.

**Proof:**
- If $\kappa$ is $n + 1$-huge, then there is $j : V^V_{j^n(\kappa)+1} \rightarrow V^V_{j^{n+1}(\kappa)+1}$.
- If $\kappa$ is $n$-huge*, then there are cardinals $\kappa = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda$ and a $\kappa$-complete normal ultrafilter $U$ over $P(\lambda)$ such that for each $i < n$,
  \[
  \{x \in P(\lambda) \mid \text{ot}(x \cap \lambda_{i+1}) = \lambda_i \} \in U.
  \]
The hierarchy of virtual large cardinals

Proposition:
- A virtually extendible cardinal is a remarkable limit of remarkable cardinals.
- If $\kappa$ is virtually huge*, then $V_\kappa$ is a model of proper class many virtually extendible cardinals.
- If $\kappa$ is virtually $n+1$-huge*, then $V_\kappa$ is a model of proper class many virtually $n$-huge* cardinals.
- If $\kappa$ is virtually rank-into-rank, then for every $n \in \omega$, $V_\kappa$ is a model of proper class many virtually $n$-huge* cardinals.
\textbf{\(\alpha\)-iterable cardinals}

\textbf{Definition:} A \textit{weak \(\kappa\)-model} (for a cardinal \(\kappa\)) is a transitive \(M \models \text{ZFC}^-\) of size \(\kappa\) and height above \(\kappa\).

Suppose \(M\) is a weak \(\kappa\)-model.

\textbf{Proposition:} TFAE.

- There exists an elementary \(j : M \rightarrow N\) with \(\text{crit}(j) = \kappa\).
- There exists an \(M\)-ultrafilter \(U\) with a well-founded ultrapower.
  - \(U\) is an \(M\)-ultrafilter if \(\langle M, \in, U \rangle \models U\) is a normal ultrafilter.
  - \(U = \{ A \in M \mid \kappa \in j(A) \}\).

\textbf{Definition:} An \(M\)-ultrafilter \(U\) is \textit{weakly amenable} if for every \(X \in M\) with \(|X|^M \leq \kappa\), \(X \cap U \in M\).

- \(U\) is partially internal to \(M\).
- Weak amenability is needed to iterate the ultrapower construction.

\textbf{Definition:} (G.) A cardinal \(\kappa\) is \textit{\(\alpha\)-iterable} \((1 \leq \alpha \leq \omega_1)\) if every \(A \subseteq \kappa\) is contained in a weak \(\kappa\)-model \(M\) for which there exists a weakly amenable \(M\)-ultrafilter on \(\kappa\) with \(\alpha\)-many well-founded iterated ultrapowers.
\(\alpha\)-iterable cardinals in the hierarchy

**Theorem:** (G., Welch, ’08)

- A 1-iterable cardinal is a limit of completely ineffable cardinals.
- An \(\alpha\)-iterable cardinal is a limit of \(\beta\)-iterable cardinals for every \(\beta < \alpha\).
- Suppose \(\alpha\) is additively indecomposable.
  - An \(\alpha\)-Erdős cardinal implies the consistency of a proper class of \(\beta\)-iterable cardinals for every \(\beta < \alpha\).
  - An \(\alpha + 1\)-iterable cardinal implies the consistency of an \(\alpha\)-Erdős cardinal.
Virtual large cardinals in the hierarchy

**Theorem:** (G., Schindler, ’15)

- A remarkable cardinal is a 1-iterable limit of 1-iterable cardinals.
- If $\kappa$ is 2-iterable, then $V_\kappa$ is a model of proper class many virtually extendible cardinals.
- A virtually $n$-huge* cardinal is an $n + 1$-iterable limit of $n + 1$-iterable cardinals.
- If $\kappa$ is $n + 2$-iterable, then $V_\kappa$ is a model of proper class many virtually $n$-huge* cardinals.
- A virtually rank-into-rank cardinal is an $\omega$-iterable limit of $\omega$-iterable cardinals.
- An $\omega + 1$-iterable cardinal implies the consistency of a virtually rank-into-rank cardinal.
Applications

Let \( C^{(n)} \) be the class club of ordinals \( \alpha \) such that \( V_\alpha \prec \Sigma_n V \).

**Definition:** (Bagaria) A cardinal \( \kappa \) is \( C^{(n)} \)-extendible if for every \( \alpha > \kappa \), there is an extendibility embedding \( j \) with \( j(k) \in C^{(n)} \).

**Vopěnka’s Principle:** For every proper class \( C \) of structures of the same type, there are \( B \neq A \), both in \( C \), such that \( B \) elementarily embeds into \( A \).

**Generic Vopěnka’s Principle:** (Bagaria, G., Schindler) For every proper class \( C \) of structures of the same type, there are \( B \neq A \), both in \( C \), such that \( B \) elementarily embeds into \( A \) in some set-forcing extension.

**Theorem:** (Bagaria, G., Schindler, ’15)

- Generic Vopěnka’s Principle for \( \Pi_1 \)-definable classes is equiconsistent with a proper class of remarkable cardinals.
- Generic Vopěnka’s Principle for \( \Pi_n \)-definable classes \( (n \geq 2) \) is equiconsistent with a proper class of virtually \( C^{(n)} \)-extendible cardinals.
Questions

**Question:** Does a virtually rank-into-rank cardinal imply the consistency of an $\omega$-Erdős cardinal?

**Question:** (vague) Can we define something akin to an algebra of embeddings for the virtually rank-into-rank cardinals?

**Question:** Applications?
Thank you!